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THE THANKSGIVING MEETING OF THE SOUTHWESTERN SECTION

The twenty-first regular meeting of the Southwestern Section was held at the University of Kansas on Saturday, December 1, 1928. The total attendance was forty-seven, including the following twenty-three members:

N. L. Anderson, Ashton, Wealthy Babcock, Florence Black, Brenke, Gaba, Garrett, Harshbarger, Hyde, Louis Ingold, Kempner, Luby, McShane, U. G. Mitchell, Mossman, O. J. Peterson, Richert, Roever, G. W. Smith, Stouffer, J. S. Turner, J. J. Wheeler, Wyant.

Professor Mitchell occupied the chair, being relieved during the morning session by Professor Roever. The morning session was devoted to the reading of the papers listed below. During the afternoon session Professor A. J. Kempner gave a special lecture by invitation of the program committee on *The development of the analytical theory of numbers in the present century.*

The titles and abstracts of the papers read are given below. The papers of Altshiller-Court, Reid, and Whyburn were read by title.

1. Professor Louis Ingold: *Generalization of the tensor laws of transformation.*

In the theory of tensors two types of transformation of sets of quantities are employed, namely, the covariant, and the contravariant laws of transformation. These involve the first derivatives of the new variables with respect to the old, or of the old with respect to the new. In connection with certain invariant expressions there occur other types of transformation involving derivatives of one set of variables with respect to the other, of order higher than the first. These are the generalizations considered in this paper.

2. Miss Nola L. Anderson: *First normal spaces in Riemannian geometry.*

If a space is defined by a general vector function of the coordinates u^i, say $f(u^1, u^2, \cdots, u^n)$, the first derivatives $f_i = \partial f/\partial u^i$ of f at each point determine the tangent space. Every vector orthogonal to the tangent space is called a normal vector. A normal vector expressible in terms of the tangent vectors f_i and their derivatives is called a first normal. It is the pur-

pose of this paper to investigate the relations connecting various first normal vectors invariant under change of coordinates. Some of these normals are independent of any loci passing through the point under consideration while others are associated with curves, surfaces or other loci passing through the point. The details are given only for two- and three-dimensional spaces.

3. Professor U. G. Mitchell: *Note on Mersenne numbers.*

At the February 1925 meeting of the Kansas Section of the Mathematical Association of America the author presented a paper summarizing the results previously obtained and the chief methods employed in factoring Mersenne and Fermat numbers. A simple method of attack based upon geometrical considerations was suggested and some results obtained by the method were also presented. In the present note is shown as a further result obtained by the method the fact that $2^{1321}-1$ has the factor 7927, proof of which is established by two independent series of congruences.

4. Professor W. C. Brenke: *On polynomial solutions of a class of linear differential equations of the second order.*

In this paper certain polynomial solutions of the differential equation $py''+qy'+r_n y=0$ are considered, when p and q are polynomials in x and r_n a function of n alone. In particular, simple formulas for the normalizing constants and generating functions are obtained, the latter by an application of the Laplace expansion as used by Darboux for the Jacobi polynomials.

5. Professor Nathan Altshiller-Court: *Some tetrahedral complexes.*

A tetrahedral complex is defined as the totality of straight lines which cut the four faces of a tetrahedron in four points having a constant anharmonic ratio. The author proves several theorems on such complexes.

6. Professor W. H. Roever: *A geometric representation of a line integral.*

By regarding the differential expression $Pdx+Qdy$ as defining surface elements in space which cut from a vertical cylindrical surface of directrix c in the xy plane lineal elements which have for unions a one-parameter family of curves on this cylindrical surface, it is possible to give a very simple geometrical representation to the value of the line integral $\int_c Pdx+Qdy$ and to see geometrically under what conditions this line integral is zero for a closed path.

7. Professor Wealthy Babcock: *On the geometry associated with certain determinants with linear elements.*

If the linear elements of the determinantal form in three variables are set equal to zero, each represents a straight line. The geometry of these lines may be studied by placing restrictions on the invariants and covariants of the linear forms which are the elements of the determinant. Certain

facts concerning the geometry of the lines which are obtained from the elements of the determinantal form of the second order and their relation to the conic, and of the lines which are obtained from the determinantal form of the third order and their relation to the plane cubic curve were brought out in this paper. From the determinantal form of the equations of plane curves, some theorems concerning the generation of plane curves of order n by projective pencils of curves of order less than n, may be readily proved.

8. Professor J. S. Turner: *The factoring of large numbers by means of positive determinants of Seelhoff type.*

This is a continuation of former papers (this Bulletin, vol. 37, pp. 163, 399). A list (probably complete) of 14 positive determinants D is given, for which: (*a*) D contains no square factor, (*b*) $T^2-D=1$, (*c*) there are two cycles of reduced forms in each genus. The method of applying these determinants to the factorization of large numbers is explained. The prime or composite character of a number of 10 digits can thus be determined in two or three hours.

9. Professor J. S. Turner: *The expression of numbers from* 12,000 *to* 30,000 *in the form* $w^3+x^3+2y^3+2z^3$.

The computation was undertaken at the instance of Professor L. E. Dickson as part of a project to verify von Sterneck's table (Wiener Sitzungsberichte, vol. 112, II*a*, (1903), pp. 1627–66) showing the number of cubes required to represent all numbers $\leqq$40,000. The method adopted is described in the American Mathematical Monthly (vol. 35 (1928), p. 113). The verification, now completed, that every number following 8041 up to 40,000 is the sum of six cubes is of the greatest importance, since this result is employed in the proof of Waring's theorem that every number is the sum of nine cubes. In addition to $p=(111111111)$ the following theorems (L. E. Dickson, American Mathematical Monthly, April, 1927, p. 177) are rendered independent of von Sterneck's table: $p=(11111112)$ (that is, Theorem III), $p=(1111114)$, where p is any positive integer $\leqq$40,000. These theorems are likewise true of the forms derived from them by partitioning 2 and 4.

10. Professor E. B. Stouffer: *A canonical expansion associated with developable surfaces.*

The author obtains a canonical expansion for the equation of a developable surface by means of the fourth order differential equation associated with the edge of regression and also determines the geometrical location of the vertices of the associated tetrahedron of reference. All the work is far more simple than when the developable surface is treated as a special case of a curved surface, the method which has been used previously.

11. Mr. W. T. Reid: *Countable differential systems containing a parameter.*

A system of integral equations of the form $y(x, \mu)=\int_0^x A(t, \mu)y(t, \mu)dt$

$+a(\mu)$ is considered, where $A(x, \mu) \equiv (A_{ij}(x, \mu))$ is a square matrix with a countable number of rows and columns, $y(x, \mu) = (y_i(x, \mu))$ and $a(\mu) \equiv (a_i(\mu))$, $(i, j = 1, 2, \cdots)$. We will call a solution of this system a set of functions $y_i(x, \mu)$, $i = 1, 2, \cdots$, each of which is absolutely continuous in x for fixed value of μ and such that $y(x, \mu)$ is a vector in Hilbert space. If (1) for each μ on L, $L_1 < \mu < L_2$, $A_{ij}(x, \mu)$ is a measurable function on X, $0 \leqq x \leqq 1$; (2) $a(\mu)$ for μ on L is a single valued vector in Hilbert space; (3) there exists a non-negative summable function $\phi(x)$ on X such that for each vector ω in Hilbert space $|A(x, \mu)\omega| \leqq \phi(x)|\omega|$; then there exists a unique solution of this system on XL. Sufficient conditions are given for the solution $y(x, \mu)$ to be continuous in (x, μ) on XL, and also to possess a partial derivative with respect to the parameter μ. The above system is also considered for each element of the matrix $A(x, \mu)$ analytic in the parameter μ.

12. Mr. W. T. Reid: *Matrix solutions of countable differential systems and their adjoints.*

If $A(x)$ is a square matrix in a countable number of dimensions limited by a summable function $\phi(x)$ on $X: a \leqq x \leqq b$, that is, for every pair of points ξ and η, of Hilbert space and for every n, $|\sum_{i,j=1}^{n} A_{ij}(x)\xi_i\eta_j| \leqq \phi(x)|\xi||\eta|$, then there exists a matrix $Y(x)$ of absolutely continuous functions satisfying equation (1): $Y'(x) = A(x) \cdot Y(x)$ "almost everywhere" on X. It is proved that if $Y(x)$ is limited at some point of X, then $Y(x)$ is limited uniformly on X. Similarly, there exists an absolutely continuous matrix $Z(x)$ satisfying the adjoint equation (2): $Z'(x) = -Z(x) \cdot A(x)$ "almost everywhere" on X, and if $Z(x)$ is limited at a point of X, then it is limited uniformly on X. If $Y(x)$ is limited and at a point of X possesses a unique limited reciprocal, then $Y(x)$ possesses on X a unique, limited, absolutely continuous reciprocal which is a solution of (2). We will call a matrix solution of (1) a limited, absolutely continuous matrix which satisfies (1) "almost everywhere" on X and has a unique reciprocal. If $Y(x)$ is a matrix solution of (1) the most general matrix solution is $Y(x) \cdot C$ where C is a constant limited matrix possessing a unique reciprocal. Furthermore, if $Z(x)$ is a matrix and $Z(x) \cdot Y(x) = C$ on X, where $Y(x)$ is a matrix solution on (1) and C is a constant matrix possessing a unique reciprocal, then $Z(x)$ is a matrix solution of (2).

13. Professor G. T. Whyburn: *On regular points of continua and regular curves of at most order n.*

This paper appears in full in the present issue of this Bulletin.

E. B. STOUFFER,
Secretary of the Section

THE THANKSGIVING MEETING IN CINCINNATI

The two hundred sixty-fourth, thirtieth regular Western, meeting of the Society was held at the University of Cincinnati on Friday and Saturday, November 30–December 1, 1928. Over one hundred persons attended the meeting, among whom were the following fifty members of the Society:

B. E. Allen, W. E. Anderson, C. L. Arnold, Bareis, Barnett, Birkhoff, Blumberg, Brand, Byrne, Coble, L. W. Cohen, H. T. Davis, P. D. Edwards, Feinler, Fry, C. A. Garabedian, Hancock, E. R. Hedrick, Archibald Henderson, Ingraham, M. M. Johnson, Kersten, Kindle, Kuhn, Latimer, LeStourgeon, MacDuffee, Metz, Michal, C. N. Moore, Mullings, T. S. Peterson, Pollard, Rainich, Rasor, C. E. Rhodes, Roth, Shohat, Sibert, W. G. Simon, Spenceley, Tripp, J. H. Weaver, Wiley, K. P. Williams, C. O. Williamson, Winston, Yanney, Yowell, Zehring.

Friday afternoon was devoted to addresses given upon invitation of the Program Committee. Professor E. R. Hedrick spoke on *Recent developments regarding non-analytic functions*. Professor Archibald Henderson spoke on *Relativity: Survey and Outlook*. These papers were followed by a lecture on *The mathematical basis of art* by Professor G. D. Birkhoff. The paper of Dr. T. C. Fry on *The use of continued fractions in the design of electrical networks* which appeared on the program on Saturday morning was also upon invitation of the Program Committee.

On Friday evening fifty-five members and their guests attended dinner in the Hotel Gibson. Professor Hancock acted as toastmaster. The toastmaster called upon Professor Birkhoff to speak of his recent trip around the world, on Dr. Fry to talk of mathematics in industry, while he himself spoke on the future of mathematical meetings in the Ohio Valley region, and called on Professors Hedrick, C. N. Moore and Blumberg to discuss the same topic. The dinner was a complimentary one for the out of town members of the Society.

On Saturday morning a resolution was passed thanking the authorities of the University of Cincinnati, the Chamber of Commerce of the City of Cincinnati, the Local Press, the Hotel Gibson and the Local Committee, of which Professor C. N. Moore was Chairman, for the excellent arrangements that had been made for the program and for the comfort and pleasure of the members of the Society. A resolution was also passed thanking Professor Hedrick, Professor Henderson, Professor Birkhoff, and Dr. Fry for their interesting and stimulating addresses.

The papers whose abstracts appear below were read as follows: papers 1–7 on Friday morning, and papers 8–17 on Saturday morning. Papers 14, 15 and 17 were read by title, the rest in person. Professor E. R. Hedrick presided Friday morning, Professor Coble presided during the addresses of Professors Hedrick and Henderson; Professor C. N. Moore presided during Professor Birkhoff's lecture; and Professors Kuhn and Birkhoff presided Saturday morning. Mr. Miller was introduced by Professor R. L. Wilder, and Mr. Churchill by Professor Rainich.

1. Professor J. A. Shohat: *On the polynomial and trigonometric approximation of measurable bounded functions, on a finite interval.*

This paper shows that the notion of the polynomial (or trigonometric) "best approximation" (in Tchebycheff sense) can be extended, *in two ways*, to the case of measurable bounded functions defined on a finite interval, making use: (a) of the upper bound, or (b) the "measurable upper bound" (C. Haskins) of such functions. The second kind of "polynomials of best approximation" is more important, for it gives an indefinitely close approximation, as in the case of continuous functions. It is also closely related to $\min \int_a^b p(x)\,|f(x)-G_n(x)\,|^m dx$, as $m\to\infty$ ($p(x)\geqq 0$, $f(x)$ are defined on (a, b), $G_n(x)$ is an arbitrary polynomial of degree $\leqq n$). Considering the same minimum for m, $n\to\infty$, we get a sequence of polynomials converging, under very general conditions, to $f(x)$ uniformly over the whole interval (a, b), with an approximation of the same order (for $n\to\infty$), as the best approximation. (See the work of G. Pólya, D. Jackson, and the author.)

2. Dr. L. W. Cohen: *On sequences of functions, of sets, and of Lebesgue integrals.*

For $f(x)\geqq 0$, we define $h(f; A)$; $H(f; A)$ as the set of all points $P(x, y)$ such that $x\subset A$, $0\leqq y<f(x)$; $0\leqq y\leqq f(x)$ respectively. Theorem I: If $f_n\geqq 0$

on A and $\lim_n f_n = f$, then (1) $h(f; a) \subset \lim \inf_n h(f_n; A) \subset \lim \inf_n H(f_n; A) \subset H(f; A)$; (2) $h(f; A) \subset \lim \sup_n h(f_n; A) \subset \lim \sup_n H(f_n; A) \subset H(f; A)$, and conversely. Let W_p be the cube $-p \leqq x_i \leqq p (i = 1, 2, \cdots, n; p = 1, 2, \cdots)$ and A_k be a set in n-space. Theorem II: If meas $A_k < +\infty$ and $\lim_p \text{meas}\,(A_k - A_k W_p) = 0$, uniformly in k, then $\lim \sup_k \text{meas}\, A_k \leqq \text{meas} \lim \sup_k A_k$. Theorem III: If $\lim_k \text{meas}\, A_k = \text{meas}\, A$ and meas $\lim \inf_k A_k = \text{meas}\, A$ then $\lim_p \text{meas}\,(A_k - A_k W_p) = 0$, uniformly in k. From I and II it follows that the usual sufficient condition $|f_n| \leqq S$, where S is summable, can be replaced by the more general condition $\lim \tau = +\infty \int_A \max(0; |f_n| - \tau) = 0$, uniformly in n, in theorems on the term-by-term integration of sequences of functions. From I and III it follows that the above condition is necessary in the case of term-by-term integration of monotonic sequences or of sequences of non-negative functions.

3. Professor H. T. Davis: *A general theory of operators with explicit application to differential equations of infinite order with polynomial coefficients of degree p.*

The present theory of operators is founded upon the Pincherle-Bourlet generatrix equation $[X \cdot F] = 1$ where the bracket symbolizes the product of the differential operators $X(x, z)$ and $F(x, z)$ in which $z = d/dx$. The case where $F(x, z)$ expands into a power series in z with constant coefficients leads to the methods of the Heaviside calculus and to formal solutions of difference equations with constant coefficients. When $F(x, z)$ expands into a series in negative powers of z the operational solution of the Volterra integral equation of the closed cycle is obtained. When the coefficients of $F(x, z)$ are polynomials of degree not greater than p the solution of the generatrix equation yields three formal series: (1) in powers of z; (2) in negative powers of x; (3) in inverse factorials in x. The regions of convergence of the solutions obtained from these formal operators are discussed.

4. Professor C. C. MacDuffee: *The discriminant matrices of a linear associative algebra.*

A sufficient condition in order that the first matrices $R_1, R_2, \cdots, R_n$ of the basal numbers of an algebra be linearly independent is found to be the non-singularity of a certain matrix called the *first discriminant matrix*, since its determinant proves to be the discriminant of the algebra as defined by Speiser. From the second matrices $S_1, S_2, \cdots, S_n$ of the basal numbers we obtain similarly a *second discriminant matrix*. These matrices are symmetric, and under a linear transformation of units are transformed like the matrix of a quadratic form. Other properties are noted.

5. Professor W. E. Roth: *On the solution of the matric equation $P(A, X) = 0$ for X commutative with A.*

The present paper takes up the matric equation $P(A, X) = 0$, where $P(\lambda, \mu)$ is a polynomial in λ and μ, and where A is a given square matrix. Solutions of the equation subject to the restriction that they be commutative with A are found. This restriction is much less severe than that im-

posed in a paper previously announced by the writer, where solutions of the same equation expressible as polynomials in the given matrix were sought. See abstract in this Bulletin, vol. 34, p. 151.

6. Mr. E. W. Miller: *The most general closed point set in n-dimensional space through which it is possible to pass an arc.*

It was shown by R. L. Moore and J. R. Kline in 1919 (Annals of Mathematics, vol. 20, pp. 218–223) that in order that a closed and bounded *plane* point set M should be a subset of an arc, it is necessary and sufficient that every component of M should be either a point or an arc, and that if t is an arc of M, no point of t, with the exception of its end points, is a limit point of $M-t$. It is shown in the present paper that if M is a closed and bounded point set in n-dimensional euclidean space, these same conditions are necessary and sufficient in order that M should be a subset of an arc.

7. Professor Henry Blumberg: *Extension and applications of the theorem on the measurable bounds of an arbitrary function.*

This theorem, communicated at the New York 1928 April meeting, is as follows: If $f(x)$—to confine the statement to the case of one variable—is an arbitrary real function, there exist two measurable functions $m_1(x)$ and $m_2(x)$ such that $m_1(x) \leqq f(x) \leqq m_2(x)$ almost everywhere, and the "metric approach," in the sense of exterior measure, via the curve $y=f(x)$ is non-infinitesimal at every point of $y=m_1(x)$ and $y=m_2(x)$. This theorem seems to be of marked utility in yielding extensions of certain properties of measurable functions to unconditioned functions. Thus the author shows that the Denjoy-Young theorems on derivatives of measurable functions carry over rather directly to general functions; likewise, for example—in a certain sense—the Arzela-Fréchet necessary and sufficient condition that an infinity of measurable functions be such as to allow the extraction of a sequence, convergent on the average. There is application, too, in the theory of integration. The theorem is extended to give information also concerning the manner of the clustering of the points of $y=f(x)$ between its measurable bounds.

8. Professor Marie M. Johnson: *Tensors of the calculus of variations.*

The tensors, considered in this paper, are connected with the non-parametric problem of the calculus of variations for a space of $n+1$ dimensions. The n expressions in the Euler differential equations, multiplied by dx, and an additional function form a covariant tensor of rank 1. An application is made of this tensor to "canonical" transformations as defined by Carathéodory. It is found that the expression which, when set equal to zero, defines transversality is an invariant. The quadratic form of the Legendre condition after it is multiplied by $1/dx$ becomes an invariant. When the Weierstrass E-function is transformed, a factor is introduced. There are $n+1$ functions which are connected with the Jacobi differential equations that form a covariant tensor. The laws of transformation of the

two determinants which are used to find the conjugate points of the Jacobi condition are discussed. In connection with fields of extremals it is shown that the expression in the Hamilton-Jacobi partial differential equation is an invariant and that the necessary conditions for an integral which is independent of the path have tensor character.

9. Professor Louis Brand: *The calculus of variations from a vector standpoint.*

This paper will appear in full in an early issue of this Bulletin.

10. Dr. T. C. Fry: *The use of continued fractions in the design of electrical networks.*

The problem in electrical circuit theory which is most frequently touched upon in textbooks is "Given a particular electrical system and a known driving force to find how the system responds." There is, however, a converse problem which is equally important, viz., "Given a known driving force and the response which it is desired to produce, to find a system which will so respond." One method of attacking this problem is through the medium of continued fractions, particularly those of the Stieltjes type. The present paper explains in a general way the part that continued fractions play in the solution of such problems and discusses some extensions which must be made in the Stieltjes theory in order to meet certain mathematical restrictions that are inherent in the technical problem.

11. Professor G. D. Birkhoff: *A theorem on the singular points of ordinary differential equations.*

In the differential equations $dx_i/dt = X_i$, $(i = 1, 2)$, in which the real analytic functions X_i vanish at the singular point (0, 0), the characteristic numbers m_1, m_2 may be either (1) real and of the same sign, (2) real and of opposite signs, or (3) conjugate imaginaries. Certain associated series converge in general cases (1) and (3), but may diverge in case (2). Nevertheless this paper establishes in this case for m_1/m_2 irrational, that by a one-to-one local transformation of the x_1, x_2 plane, in which the functions involved are continuous together with all their partial derivatives, analytic except along the two invariant analytic curves through (0, 0) and possessed of certain other properties, the differential equations reduce to $dx_i/dt = m_i x_i$, $(i = 1, 2)$. This is essentially the same normal form previously obtained in the other two general cases by means of a one-to-one analytic transformation based upon the convergent series available in these cases.

12. Professor G. Y. Rainich: *Plane waves in curved space-time.*

The group of Lorentz transformations is found to which the electromagnetic field of a plane wave belongs, and a metric is sought which admits an isomorphic group. Using curvilinear coordinates introduced before, the general symmetric tensor of rank two admitting the group is found: it is taken as the fundamental tensor of a metric and subjected to the condition

that the contracted Riemann tensor of that metric has the form of an electromagnetic stress energy tensor. The consequence of this restriction is that the space has zero curvature. This result is interpreted as indicating that in general relativity plane waves do not exist—a conclusion arrived at by Baldwin and Jeffery in a different way. It follows that the only type of fields in curved space-time which possess symmetry properties of radiation are fields with plane singularities. It is indicated that a field with a great number of such singularities may be considered as having the symmetry of a plane wave if local discontinuities are disregarded much as a part of space with many point singularities corresponding to material particles may be regarded as a continuous body by neglecting local discontinuities.

13. Mr. R. V. Churchill: *On the geometry of the Riemann tensor.*

At any point of a four-dimensional Riemann space a tensor of the second type is determined by a skeleton (that is, a set of four mutually perpendicular directions intrinsically related to the tensor) and three numbers. Moreover the skeleton is uniquely determined by this tensor. A tensor of the first type is determined by a skeleton and five numbers, but the skeleton is not uniquely determined by this tensor. From any one skeleton and its set of five numbers five other skeletons with their corresponding sets of five numbers can be found. Each of these six skeletons with its set of numbers serves in the same way in the determination of their tensor. It has been shown that the Riemann tensor at any point is the sum of a tensor of the first type and a tensor of the second type. Hence the Riemann tensor at a point of a four-dimensional Riemann space is determined by two skeletons and eight numbers. In this paper most of these results are obtained by using six-vectors for the arguments of the Riemann tensor after a restatement and extension of the theory of six-vectors is made.

14. Dr. I. M. Sheffer: *On the theory of sum-orthogonal polynomials.*

Let there be given two sets of real numbers: $\{p_k\}$, $\{x_k\}$, $k=0, 1, \cdots, \infty$, with $p_k>0$ and $x_i \neq x_j$ for $i \neq j$. We denote by $S[u(x)]$ (when it exists) the weighted sum for $u(x)$ relative to $\{p_k\}$ over the range $\{x_k\}$: $S[u(x)] = \sum_{k=0}^{\infty} P_k u(x_k)$. Two functions $u(x)$, $v(x)$ are defined to be sum-orthogonal if $S[uv]=0$. This paper is concerned with the existence and properties of a (unique) system of polynomials $\{T_n(x)\}$ which are normal and sum-orthogonal.

15. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem and continuous curves in n-space.*

The author proves that a continuous curve (bounded) in n-space without a cut point which satisfies the Janiszewski-Mullikin theorem must be the simple sphere of analysis situs, establishing in a sense a "converse theorem." Using generalizations of certain plane theorems he is able to discard the condition of no cut point and to discuss in full the most general solution.

16. Professor K. P. Williams: *A generalization of the Cauchy-Riemann equations.*

The sum, product, and quotient of two analytic functions are analytic. This shows that there is a formal connection between the rules for differentiating the product and quotient of two functions and the Cauchy-Riemann equations, and suggests the following problem considered in this paper. Let $u(x, y)$, $v(x, y)$ and $u^*(x, y)$, $v^*(x, y)$ be two solutions of $au_x+bu_y+cv_x+dv_y=0$ where a, b, c, d are constants. Let it be required to annex other conditions such that U and V will also satisfy the given and annexed conditions, where $U+iV$ is the sum, product, or quotient of $u+iv$ and u^*+iv^*. It is found that one is led to the system of equations $Au_x+u_y+v_x=0$, $u_x-Av_x-v_y=0$. These equations can be regarded as a natural generalization of the Cauchy-Riemann equations. It is found that $u+iv$ can be considered as an analytic function of $(x+iv)-Ay$.

17. Professor A. D. Campbell: *Note on the Plücker equations for plane algebraic curves in the Galois fields.*

This paper appears in full in the November-December 1928 issue of this Bulletin.

M. H. INGRAHAM,
Associate Secretary

THE ANNUAL MEETING IN NEW YORK

The thirty-fifth Annual Meeting of the American Mathematical Society was held in New York City, from Wednesday to Monday, December 26–31, 1928, in conjunction with the meetings of the American Association for the Advancement of Science and the Mathematical Association of America. All the sessions of the Society were held at Columbia University: the Gibbs Lecture in the Casa Italiana, the Symposium on Quantum Mechanics in Room 401 of the Physics Laboratories, all other sessions in Barnard College. The meeting opened on Wednesday evening with a session of the Board of Trustees at the Hotel Lincoln (the headquarters of the mathematical organizations). Thursday was devoted to sectional sessions of the Society. The third award of the Bôcher Memorial Prize was announced at a general session on Friday forenoon; this session was followed by the annual business meeting, at which trustees, officers and members of the Council were elected. The first part of Friday afternoon was devoted to a joint session of the Society, the Mathematical Association and Section A of the American Association for the Advancement of Science; this session was followed by the sixth Josiah Willard Gibbs Lecture under the joint auspices of the Society and the A.A.A.S. On Saturday morning the Society held a joint session with Section K of the A.A.A.S. (Social and Economic Sciences); the Mathematical Association also held its sessions on Saturday. Monday was devoted to a Symposium on Quantum Mechanics, held under the joint auspices of the American Mathematical Society and the American Physical Society.

The joint dinner of the mathematical organizations, held on Friday evening in the Men's Faculty Club of Columbia University, was attended by about one hundred ninety persons. Professor Dunham Jackson was toastmaster.

The mathematicians and their guests were invited to the

various general lectures and receptions held by the A.A.A.S. They enjoyed especially a complimentary concert given to the attending scientists in Carnegie Hall on Sunday afternoon by the New York Philharmonic-Symphony Society.

The attendance at this meeting (the largest on record) included the following two hundred eighty-two members of the Society:

C. R. Adams, R. B. Adams, Adshead, A. A. Albert, E. S. Allen, Joseph Allen, Altshiller-Court, R. L. Anderson, L. B. Andrews, R. C. Archibald, R. G. Archibald, C. L. Arnold, H. E. Arnold, Ashcraft, Atchison, R. W. Babcock, Bacon, Bamforth, Barnett, Beetle, A. A. Bennett, Theodore Bennett, A. C. Berry, William Johnston Berry, Birkhoff, Blumenthal, Bowden, Bower, Bradshaw, Brinkmann, Bristol, Burington, Burkett, Cairns, Caldwell, B. H. Camp, A. D. Campbell, Carlen, R. D. Carmichael, E. T. Carroll, Carruth, A. G. Clark, Clarke, L. W. Cohen, Congdon, Coolidge, Cope, A. H. Copeland, L. P. Copeland, Cramlet, Currier, Dadourian, Darkow, J. E. Davis, Decker, Dehn, Denton, Dimick, DoBell, Dorwart, Dorweiler, Dresden, Edmondson, Eisenhart, W. W. Elliott, Engstrom, H. P. Evans, Farnum, Feinler, Fiske, Fite, D. A. Flanders, Focke, Fort, A. L. Foster, M. C. Foster, Philip Franklin, Frink, Frumveller, Fry, C. A. Garabedian, Gehman, Gergen, J. L. Gibson, Gill, D. C. Gillespie, Gilman, Glasgow, J. W. Glover, P. H. Graham, A. A. Grant, Gronwall, C. C. Grove, Hall, Hallett, Hammatt, Hansell, J. G. Hardy, E. R. Hedrick, Robert Henderson, Hickson, Hille, Himwich, Hofmann, Hollcroft, J. C. Hughes, O. M. Hughes, Huntington, Dunham Jackson, Jeffery, M. I. Johnson, R. A. Johnson, Jonah, E. H. Jones, Kasner, Kazarinoff, O. D. Kellogg, Keyser, Kholodovsky, B. F. Kimball, Rose Klein, Kline, Kramer, Lambert, Landers, Landis, Langman, Larew, Lefschetz, D. H. Lehmer, Lehr, Leib, Leinbach, Lepeshkin, Harry Levy, F. P. Lewis, Linehan, Ling, Littauer, Longley, Lotka, Lubben, Lubin, McGiffert, MacColl, MacCreadie, MacInnes, Marden, Mathewson, S. P. Mead, Mears, Meder, J. E. Merrill, E. J. Miles, Norman Miller, H. H. Mitchell, Mode, Molina, C. N. Moore, R. L. Moore, Moriarty, Frank Morley, C. C. Morris, Richard Morris, D. S. Morse, Marston Morse, Morton, Murnaghan, F. H. Murray, Nassau, C. A. Nelson, Newman, Ogburn, Olson, Oppenheim, Ore, Osgood, F. W. Owens, H. B. Owens, Paradiso, Patterson, Pattillo, Paxton, Pell-Wheeler, F. W. Perkins, T. S. Peterson, Pfeiffer, H. B. Phillips, Pierpont, Pixley, Pooler, Poritsky, Post, R. G. Putnam, Rainich, Rambo, Ramsdell, Raudenbush, Rawles, Reddick, C. J. Rees, M. S. Rees, R. G. D. Richardson, D. E. Richmond, Rider, Risselman, Ritt, Robertson, Robison, E. D. Roe, Roever, Roman, Roos, Rorer, J. E. Rowe, Ruger, Rupp, Saurel, Sauté, Schoonmaker, Seely, Seidlin, Serghiesco, Shaub, Sheffer, Shewhart, Siceloff, Silverman, Simons, Singer, Slaught, Slobin, Slotnick, Smail, A. H. Smith, C. E. Smith, W. M. Smith, Stafford, M. E. Stark, Stetson, Stocker, Stokes, M. H. Stone, Struik, Tamarkin, J. M. Thomas, J. E. Thompson, L. T. E. Thompson, Trjitzinsky, B. M. Turner, Tyler, Vallarta, Vandiver, Van Vleck,

Waddell, Wahlert, G. W. Walker, C. M. Walsh, J. L. Walsh, H. E. Webb, Wedderburn, D. W. Weeks, Weida, Weisner, M. E. Wells, Weyl, A. M. Whelan, H. S. White, Whited, Widder, Wiener, C. E. Wilder, F. B. Williams, W. L. G. Williams, John Williamson, A. H. Wilson, E. B. Wilson, E. W. Wilson, W. A. Wilson, J. W. Young, Margaret M. Young, Zariski.

At the meeting of the Council, the following fourteen persons were elected to membership in the Society:

Mr. John Geoffrey Adshead, Dalhousie University;
Professor Herbert Eli Arnold, Wesleyan University;
Professor May Margaret Beenken, State Teachers College, Oshkosh;
Professor Ludwig Bieberbach, University of Berlin;
Mr. Leo Branovan, University of Minnesota;
Professor James Edward Donahue, University of Vermont;
Professor Karl Wilmot Folley, Trinity College;
Miss Rose Klein, Washington Irving High School;
Miss Erma M. Leinbach, Sweet Briar College;
Mr. Allen Shelly McMaster, University of Colorado;
Dr. Edward Aaron Saibel, University of Minnesota;
Mr. Howard Elmer Wahlert, Washington Square College, New York University;
Miss Dorothy Walcott Weeks, Wellesley College;
Professor Hermann Weyl, Princeton University.

The following were elected to membership as nominees of Allyn and Bacon:

Mr. Hubert Banks Huntley, Brown University;
Mr. William Clemmer Mitchell, University of Cincinnati;
Mr. George Sauté, Harvard University;

as nominees of the National Life Insurance Company of the United States of America:

Mr. Aubrey Wilfred Landers, Brown University;
Dr. William Maier, University of Chicago;

as nominees of the University of Michigan:

Mr. Ruel V. Churchill, Mr. Shirley E. Field, Mr. Nevin C. Fisk, Mr. Walter O. Menge, Mr. Theodore E. Raiford, all of the University of Michigan;

as nominees of the Westinghouse Electric and Manufacturing Company:

Mr. Charles LeG. Fortescue, Mr. Clinton R. Hanna, Mr. George R. Karelitz, Mr. John F. Peters, Mr. Nicholas Rashevsky, Mr. Morris Stone, Mr. Demetry I. Vinogradoff, all of the Westinghouse Company's Engineering Department at East Pittsburgh.

The ordinary membership in the Society is now 1796, in-

cluding 178 nominees of sustaining members and 80 life members. There are also 38 sustaining members. The total attendance of members at all meetings during the past year was 769; the number of papers read was 439. The number of members attending at least one meeting was 560. At the annual election 265 votes were cast.

The reports of the Treasurer and of the auditors (Professors H. W. Reddick and Paul Saurel) showed a balance of $4709.55 exclusive of the balances in the Bulletin, Transactions, Colloquium, Journal and special funds, and of the life membership reserve. The Society's Endowment Fund now has securities of par value $75,000 yielding an annual income of $3515; sustaining membership fees for the year amounted to $4810. The amount received from sales of the Society's publications was $6776.38. The trustees adopted a budget for 1929 showing estimated expenditures and receipts as $32,245.42 and $30,855.42 respectively. During 1928 members made special contributions, amounting, approximately, to twelve hundred dollars, which wiped out the deficit on the year's business; it is hoped that these contributions will be increased this year so that the deficit anticipated for 1929 may be covered.

The Librarian reported that the Library of the Society now contains 7773 volumes.

At the annual election, which closed on Friday morning, the following trustees and officers and other members of the Council were chosen:

Board of Trustees, Professors J. L. Coolidge, W. B. Fite, E. R. Hedrick, R. G. D. Richardson, Oswald Veblen.

President, Professor E. R. Hedrick.

Vice-President, Professor Solomon Lefschetz.

Associate Secretary, Professor Arnold Dresden.

Member of the Editorial Committee of the Bulletin, Professor W. R. Longley.

Member of the Editorial Committee of the Transactions, Professor Dunham Jackson.

Members of the Editorial Committee of the Colloquium Publi-

cations, Professors G. D. Birkhoff, R. L. Moore, J. W. Young.

Members of the Council, Professors H. F. Blichfeldt, Daniel Buchanan, B. H. Camp, J. R. Kline, E. P. Lane.

The tellers appointed to count the ballots were Professors C. R. Adams and R. A. Johnson.

At the annual business meeting of the Society on Friday morning Vice-President Keilogg presented the following resolution, which was unanimously adopted: The American Mathematical Society desires to express to the President and Trustees of Columbia University, its thanks for their hospitality in connection with the present meeting, and its appreciation of the effective measures taken to make the meeting successful.

The following resolution in memory of the late Dean Fine, also presented by Vice-President Kellogg, was unanimously adopted by the Society:

The members of the American Mathematical Society have learned with profound sorrow of the death of Dean Henry Burchard Fine on Saturday last, December 22. Dean Fine was a member of this Society from its earliest days; he was its president in 1911 and 1912, and a member of its Council for many years thereafter. His whole life was devoted with rare singleness of purpose to the cause to which this Society is dedicated, the building up of mathematical science in America. He was an inspiring teacher; his influence was undeviatingly exerted for rigorous thinking and for the highest standards of scientific honesty and quality of production. These things he fostered not only by the force of his personality but also by his keen appreciation of talent in others and his selfless furthering of the opportunities and careers of those so endowed. The department of mathematics at Princeton is a monument to the forty-nine years of patient and wise planning which Dean Fine devoted to its building. The Society desires to express its sense of great loss in the death of its honored member, and order that this minute be entered in its records, also that copies be sent, through the President, to Princeton University, and to his daughter, his brother, and to his sisters.

The following appointments were reported: to represent the Society at the inauguration of President H. N. Davis, of the Stevens Institute of Technology, on November 23, 1928, Vice-President O. D. Kellogg; to represent the Society at the American Mining Congress on December 5–8, 1928, Dr. C. E. Van Orstrand.

At the meeting of the Board of Trustees it was decided to raise the prices of back volumes of the Transactions and the Bulletin and, effective January 1, 1930, to raise the price of current volumes of the Transactions to $8.

At the meeting of the Council, Professor James Pierpont was named to succeed President Virgil Snyder as one of the three representatives of the Society in the Division of Physical Sciences of the National Research Council. Professors E. V. Huntington, chairman, Dunham Jackson and H. E. Slaught were appointed as a committee to advise the printers as to forms of mathematical type.

It was voted to thank Professor Tomlinson Fort for his services on the Committee on Sustaining Memberships.

The Council decided to grant the request of the San Francisco Section that the meetings on the Pacific Slope be hereafter designated as regular meetings of the Society.

Professors E. B. Stouffer, chairman, M. H. Ingraham and H. L. Rietz were appointed a Committee on Program for the Western Meetings of the Society.

It was announced that the seventh Josiah Willard Gibbs Lecture would be held at Des Moines in December 1929 in connection with the Western meeting of the Society and the sessions of the American Association for the Advancement of Science and that a committee consisting of Professors O. D. Kellogg, chairman, E. B. Stouffer and Warren Weaver had been appointed to recommend a lecturer to the Council.

It was announced that the title of Professor R. L. Moore's colloquium lectures, to be delivered in Boulder at the Summer Meeting of 1929, will be *Point set theory*, and that Professor Lefschetz will deliver the colloquium lectures, at

Brown University at the summer meeting of 1930, the title to be announced later.

The general session on Friday morning opened with an address by Professor W. F. Osgood on *Maxime Bôcher*. Immediately afterwards the chairman, Professor E. B. Van Vleck announced that the Council, upon recommendation of a committee consisting of Professors Bell, Carmichael and Lefschetz, had made the third award of the Bôcher Memorial Prize to Professor J. W. Alexander, of Princeton University, for his memoir entitled *Combinatorial analysis situs*, published in volume 28 of the Transactions of this Society. Professor Alexander was unfortunately prevented by illness from being present to receive the award. The session closed with an invited address by Professor James Pierpont, entitled *On the motion of a rigid body in a space of constant curvature.*

At the joint session of the Society, the Mathematical Association, and Section A, held on Friday afternoon, the following papers were read:

I. *The relation of statistics to modern mathematical research*, by Professor Dunham Jackson, retiring Vice-President of Section A. This paper was published in Science for January 18, 1929.

II. *The heroic age of geometry*, by Professor J. L. Coolidge. This address, delivered upon invitation of the Society and the Mathematical Association, was published in this Bulletin, for January-February, 1929.

The sixth Josiah Willard Gibbs Lecture, entitled *An introduction to the theory of numbers*, was to have been delivered by Professor G. H. Hardy, of the University of Oxford. Shortly before the hour announced for this lecture it was learned that illness would make it impossible for Professor Hardy to be present. Professor H. W. Brinkmann, of Harvard University, very graciously consented to present the material of the lecture. The lecture will be printed in full in an early issue of this Bulletin.

The attendance at the joint session and at the Gibbs Lecture ran well over 300.

On Monday was held a symposium on quantum mechanics, under the joint auspices of the American Mathematical Society and the American Physical Society. This was in the form of invited addresses, each followed by discussion. The program was as follows:

Physical meaning of wave mechanics, by Professor J. C. Slater. Discussion led by Dr. W. F. G. Swann and Professor E. U. Condon.

Statistical interpretation of various formulations of quantum mechanics, by Professor J. H. Van Vleck. Discussion led by Professors E. H. Kennard, D. J. Struik, and E. C. Kemble.

The problem of symmetry in quantum mechanics, by Professor Hermann Weyl. Discussion led by Professors G. Y. Rainich, S. A. Goudsmit, and H. W. Brinkmann.

Harmonic analysis and quantum mechanics, by Professor Norbert Wiener. Discussion led by Dr. T. H. Gronwall and Professors H. P. Robertson and G. E. Uhlenbeck.

More than 400 persons, a much larger number than had been anticipated, including chemists as well as mathematicians and physicists, were present at the sessions of this symposium.

Titles and abstracts of the papers read at the regular sessions of the Society follow below. The presiding officers were as follows: Vice-President Kellogg over Section I (Analysis) and (relieved by Professor Lefschetz) over Section III (Analysis and Applications); Vice-President Young over Section II (Geometry) and Section IV (Algebra and Analysis Situs); ex-President Van Vleck at the general session on Friday morning; Professor R. C. Archibald at the joint session with the Mathematical Association and Section A; ex-President Birkhoff during the Gibbs Lecture and (relieved by Dr. T. C. Fry) at the afternoon session of the Symposium on Quantum Mechanics; Professor E. V. Huntington at the joint session with Section K; and Professor K. T. Compton at the morning session of the Symposium. The papers numbered 1–21 were read before Section

I, Nos. 22–39 before Section II, Nos. 40–63 before Section III, Nos. 64–92 before Section IV, No. 93 at the general session, and Nos. 94–99 at the joint session with Section K. The papers of the following authors were read by title: Albert (first, second, and fourth papers), Altshiller-Court (second paper), Ayres, Barrow, Berry, Brinkmann (second and third papers), Douglas, Ettlinger, Frink (third paper), Garver, Harkin, Hickey, Keller, Littauer, Lubben (first paper), McGiffert, Poritsky (last three papers), Powers, Reynolds, Roberts, Rutledge, Schwatt, Sheffer (first paper), Stone (first three papers), Tamarkin (first and second papers), Sullivan, G. T. Whyburn, W. M. Whyburn, and Widder. Professor Hagihara was introduced by Professor Birkhoff, Mr. Muckenhoupt by Professor Wiener, and Dr. Vasilesco by Professor Kellogg.

1. Professor H. J. Ettlinger: *Expansion problems for general second-order systems.*

Expansion problems are considered for the system $y'=K(x,n)z$, $z'=G(x,n)y$, with self-adjoint boundary conditions. The method used is that of transforming to polar coordinates, $y=u\cos v$, $z=u\sin v$. Asymptotic expansions are obtained for characteristic functions and characteristic numbers. Expansion theorems are obtained for a pair of arbitrary functions. The results contain the work of Camp, Hurwitz, and others as special cases.

2. Dr. I. M. Sheffer (National Research Fellow): *On the properties of systems of polynomials which are functionally orthogonal.*

Let there be given a linear operator L which assigns a number to each member $u(x)$ of a suitable class of functions defined over a certain range R of points: $L[u(x)]=c$. Two simple examples of L are (1) $L[u]=\int_a^b p(x)u(x)dx$, where $p(x)$ is a given function; (2) $L[u]=\sum_{k=0}^{\infty}p_k u(x_k)$, where the sequence $\{p_k\}$ is given. In (1) the range R is the set of points $a\leqq x\leqq b$; in (2) R is the set of points $x_0, x_1, x_2, \cdots$. The functions $u(x)$ and $v(x)$ are said to be L-orthogonal if $L[u(x)\cdot v(x)]=0$. We consider a system of polynomials $\{T_n(x)\}$ such that $L[T_m(x)T_n(x)]=0$, $m\neq n$, and examine the properties of this system. These properties are found to be essentially inherent in the *linear character of the operator.*

3. Dr. I. M. Sheffer: *On systems of polynomials which are permutable.*

The Appell polynomials have the property of permutability: that is, if $\{P_n(x)\}$, $\{Q_n(x)\}$ are any two sets of Appell polynomials, they are

permutable. Appell polynomials form, then, a class of permutable sets of polynomials. But this is not the only class having this property. We obtain further such classes, and examine the general theory. To each set of polynomials we assign a characteristic equation which the set satisfies. The theory of permutability links up, on one side, with the theory of summability by the method of Toeplitz and Hurwitz-Silverman, and, on the other side, with the condition that certain linear operators be themselves permutable.

4. Dr. Irwin Roman: *A formula for the summation of series.*

Taylor's theorem leads to the formula $f(n)=Z(n+h)-Z(n-h)$, where $Z(x)=(2h)^{-1}\left[\int f(x)dx+\sum_{u=1}^{\infty}b_{2u}h^{2u}f^{(2u-1)}(x)\right]$. Hence we may write the formula $\sum_{n=t}^{s}f(k+2hn)=Z[k+(2s+1)h]-Z[k+(2t-1)h]$, and also the formula $\int_{n-a}^{n+b}f(x)dx=2hf(n)-\sum_{u=1}^{\infty}b_{2u}h^{2u}[f^{(2u-1)}(n+h)-f^{(2u-1)}(n-h)]$. The constants b_{2u} are determined by a recursion formula. They are expressible in terms of the Bernoulli numbers and conversely. Illustrative examples indicate the wide applicability of the formulas, but no attempt is made to discuss the conditions necessary for the formulas to be applicable to a particular problem.

5. Professor J. D. Tamarkin: *On Volterra-Stieltjes integral equations.*

In this paper a system of n integral equations is considered of the type $Y(x)=F(x)+\int_a^x d_\xi K(x,\xi)Y(\xi)$, where (in matrix notation) the vector $F(x)$ and the matrix $K(x,\xi)$ are given. It is assumed that $F(x)$ is bounded and Borel measurable and $K(x,\xi)$ is bounded on $a\leqq\xi\leqq x\leqq b$, Borel measurable in x for fixed ξ and continuous and of bounded variation in ξ for fixed x, the total variation being uniformly bounded. The uniqueness and existence of a solution (bounded and Borel measurable) is proved. Certain boundary problems of Volterra-Stieltjes type are considered, and the notion of the Green's function for such problems introduced. The results obtained are applied to the discussion of the generalized adjoint problem.

6. Professor J. D. Tamarkin: *Expansion of an arbitrary function in fundamental functions of a boundary value problem of integral type.*

It is shown, by means of some results previously obtained by the author, that if the "boundary" conditions contain integral terms, then the set of fundamental functions in general is not closed, and a set of supplementary conditions must be satisfied by the function which has to be expanded. However, the set of fundamental functions can always be extended to a closed one by addition of a finite number of functions which are easily determined.

7. Professors Einar Hille and J. D. Tamarkin: *On the summability of Fourier series.* Note 2.

The authors have continued their work on the summability of Fourier series and have found sufficient conditions for the applicability of Hausdorff

means for almost all values of the variable. A more detailed outline of the results will appear in a forthcoming number of the Proceedings of the National Academy of Sciences.

8. Dr. F. W. Perkins: *On the theory of mediation.*

By the mediant of a function $F_0(P)$ (where P is a point of a bounded open region T in a 3-dimensional euclidean space) is meant the function $F_1(P)$ whose value at P is equal to the average value of $F_0(P)$ on the surface of the largest sphere with center at P containing in its interior only interior points of T. Another form of mediation may be obtained by using the average with respect to volume. The definition may also be generalized for a region T in an n-dimensional euclidean space. In this paper a number of general properties of mediation are obtained, and especial attention is given to the discussion of the sequence $F_0(P)$, $F_1(P)$, $F_2(P)$, $\cdots$, obtained by successive application of the process. The study of this sequence shows that mediation is closely connected with the theory of harmonic functions, particularly with the first boundary value problem of potential theory. This phase of the theory has been investigated by Lebesgue (Comptes Rendus, vol. 154 (1912), p. 355), and the present author (Comptes Rendus, vol. 184 (1927), p. 182).

9. Professor I. J. Schwatt: *The summation of series the terms of which alternate in sign, not term-by-term but by groups of terms.*

The author has considered the sum of like powers of the first n natural numbers, the terms alternating in sign by groups of terms, so that, say, h positive terms are followed by h negative terms, and also the case where h numbers of terms in the positive and negative groups are unequal. The results involve the Bernoulli numbers.

10. Professor I. A. Barnett: *A characteristic property of conformal transformations in function space.*

Let R_x denote the totality of all continuous functions $f(x)$; and R_x' the totality of all functions $[f(x), \sigma]$, where σ is a real continuous variable. The projective group in the space R_x' is defined and explicit formulas obtained for all the infinitesimal transformations in this space. The author then finds the subgroup all of whose transformations leave the manifold $\int f^2(x)dx+\sigma^2=1$ invariant. Calling these transformations S, he proves that if S are subjected to a stereographic projection in the space R_x on the linear spread $\sigma=$const., the resulting transformations are the same as those that leave invariant the angle between any two curves in R_x, viz., the conformal group in R_x. This is a generalization of a well known property in spaces of a finite number of dimensions.

11. Professor M. H. Stone: *Expansions associated with self-adjoint elliptic partial differential equations.*

In the present paper the classical problem of the flow of heat in a non-homogeneous conducting medium with assigned initial temperature and

permanently assigned boundary temperature is considered from a modern point of view. The solution of the differential equation of parabolic type governing the flow is studied as a functional of the boundary conditions. There results an important theorem concerning the representation of an arbitrary bounded measurable function in terms of the solutions of an elliptic partial differential equation involving a parameter.

12. Professor M. H. Stone: *Theory of Hermitian transforms.*

In a previous communication to the Society, the author has considered the general theory of Hermitian symmetric operators in abstract Hilbert space. Operators of this type which satisfy the condition that some iterate is the identity are studied in the present paper. It is found that if T is such an operator either T or T^2 is the identity. The second alternative is given special consideration. A useful sufficient condition that an operator be of this character is obtained, and is then applied to yield a unified treatment of certain familiar operators of analysis, including the Fourier and Hankel transforms.

13. Professor M. H. Stone: *Gauss' law of error in several variables.*

In the Mathematische Zeitschrift for 1923, Pólya gave a derivation of Gauss' law of error from a functional equation. The present paper extends the derivation to the law of error for several variables. The method is similar to that of Pólya, and the results differ very little from his. With the increase of the number of variables from one to several, it is necessary to increase the number of hypotheses concerning the law of error in order to determine it uniquely. The relation of these hypotheses to the classical assumptions of Laplace and of Gauss is discussed.

14. Professor M. H. Stone: *Unitary transformations in abstract Hilbert space.*

In this paper the following theorem is proved: every one-parameter continuous group of unitary transformations in abstract Hilbert space possesses a generating infinitesimal transformation of the form iT, where T is an Hermitian symmetric operator which is maximal, and conversely.

15. Professor G. M. Robison: *The consistency and equivalence of certain definitions of summability of divergent double series.*

This paper deals with definitions of summability of divergent double sequences which can be expressed by means of a linear transformation. Let $x_{m,n}$ be the given sequence. A new sequence $y_{m,n}$ is defined by the relation $y_{m,n}=\sum_{k=1,\,l=1}^{m,\,n} a_{m,n,k,l}x_{k,l}$, where the $a_{m,n,k,l}$'s form a set of constants defining the transformation, which we denote by (A). If $\lim_{m,n\to\infty} y_{m,n}$ exists and equals λ, we say that λ is the generalized value of the sequence $(x_{m,n})$ by the transformation (A). This paper shows that all regular transformations of a certain class are consistent and gives a criterion for the equivalence of two regular definitions of summability.

16. Dr. Stefan Serghiesco: *On the number of multiple common roots of a system of n equations.*

In several earlier papers, the author has introduced a new method for obtaining the number of simple common zeros of a set of n functions, and has given two new general formulas. In the present paper, the author extends this method of the case of multiple common zeros, stating the conditions of possibility with the corresponding formulas.

17. Dr. T. H. Rawles: *On minimizing a function of two definite integrals.*

The problem discussed in this paper is that of minimizing a function of two definite integrals of the form $U(\int_{x_1}^{x_2} f(x, y, y')dx, \int_{x_1}^{x_2} g(x, y, y')dx)$. Besides obtaining the extremals, further necessary conditions are developed analogous to those of Legendre and Jacobi in the simplest problem in the calculus of variations. Sufficient conditions are obtained by means of an invariant integral.

18. Professor George Rutledge: *Limiting values of Lagrangean coefficients.*

The coefficients, C_j, of the Lagrange polynomial, of degree $2n$, $p^{[2n]}(\lambda) = C_{-n}y_{-n} + \cdots + C_0y_0 + \cdots + C_ny_n$, useful for interpolation without differences, approach limiting values as n becomes infinite, as follows: $\lim C_j = (-1)^{j+1}[\lambda/(j-\lambda)](\sin \lambda\pi)/(\lambda\pi)$.

19. Professor George Rutledge: *Fundamental tables for Lagrangean coefficients.*

This paper gives tables with fifteen decimal places from which the coefficients of the Lagrange polynomial are readily derived for any range of λ and for any even degree up to and including sixteen.

20. Professor D. F. Barrow: *Functions of iteration and their relation to non-linear difference equations.*

The symbol $f^n(t)$ denotes n iterations of the given function $f(t)$, and defines a function of two arguments; t, a continuous variable, and n, an integer. The author proposes to define this *function of iteration* for all values of n; and this is, by definition, made tantamount to finding analytic solutions of the difference equation $\phi(x+1) = f[\phi(x)]$. The solution of the problem is embodied in a group of existence theorems and approximation formulas for numerical computation. The chief interest centers about the *points of equilibrium* which satisfy $A = f(A)$. The paper closes with a number of examples, including the analytic iteration of the sine and exponential functions, that is, the solution of the equations $\phi(x+1) = e^{\phi(x)}$, and $\phi(x+1) = \sin \phi(x)$.

21. Professor D. V. Widder: *An inequality related to one of Hilbert.*

In the Mathematische Zeitschrift of 1921, L. Fejér and F. Riesz established a fundamental inequality, and applied it to the proof of a familiar inequality of Hilbert. They proved that if $f(z)$ is analytic in a circle $|z| \leqq 1$, then $\int_{-1}^{1} |f(z)|^2 dz \leqq \frac{1}{2}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta$. In the present paper, the circle is replaced by a half plane, and then the inequality $\int_0^\infty |f(z)|^2 dz \leqq \frac{1}{2}\int_{-\infty}^{\infty} |f(iy)|^2 dy$ is established under suitable conditions. This result is used to prove the inequality $\sum_{m,n=0}^{n} a_m a_n/(m+n+1) \leqq \pi \sum_{m,n=0}^{N} (m+n)!\, a_m a_n \cdot (m!n!2^{m+n+1})^{-1}$, where the a_m are any real numbers. This inequality includes that of Hilbert, for it is shown that the right hand member is not greater than $\pi \sum_{n=0}^{N} a_n^2$.

22. Professor C. A. Nelson: *On surfaces possessing a net of plane isothermally-conjugate curves.*

It is well known that the Laplace transforms of a surface relative to a conjugate system of plane curves are developable. This paper studies the effect of the isothermal property upon such surfaces and their projective properties are determined. The problem of integrating the defining system of partial differential equations is investigated, and, in certain cases, the complete integration is effected, so that explicit formulas are obtained for the coordinates of the surface.

23. Professor C. T. Sullivan: *The determination of plane nets characterized by certain properties of their Laplace transforms.*

This paper will appear in full in an early issue of this Bulletin.

24. Dr. M. M. Slotnick: *A conjugate net of special type.*

A conjugate net for which each tangent to the v-curve of its first Laplace transform intersects the tangent to the corresponding u-curve of its minus first Laplace transform will be called a net Δ. The point of intersection of these tangents is the pivotal point of the axis. A net whose ray curves or whose axis curves are indeterminate is a net Δ. A characteristic property of nets Δ is the following: the tangents to the axis curves and to the ray curves separate the tangents to the curves of the net harmonically. If a net Δ has equal point [tangential] invariants, the ray [axis] curves constitute the associate conjugate net of the given net.

25. Professor Edward Kasner: *Higher partial derivatives of polygenic functions of several variables.*

The higher partial derivatives of polygenic functions of two or more complex variables depend, not only on the curves of approach for each of the independent variables (slope, curvature, etc.), but also on the correspondence between the curves. Also, as indicated in earlier papers (see Transactions of this Society, vol. 30, p. 818) we must take into account the order of the variables.

26. Professor T. R. Hollcroft: *The variety of order n in n dimensions.*

There is a one-to-one correspondence between varieties of order n in i dimensions and varieties of order i in n dimensions, since the number of conditions necessary to determine each is the same. In this correspondence, the variety for which $n=i$ is self corresponding. It is proved that every general variety of order n in n dimensions, $n\geqq 3$, contains the following: an $(n-3)$-fold infinity of lines; a triple infinity of plane elliptic cubics; a single infinity of conics; a finite number of sets of three coplanar lines; for $n\geqq 4$, a $(d+1)(d-3)$-fold infinity of sets of $d+1$ lines, each set contained in a linear d-space, $d<n$; an $(n-1)$-fold infinity of space cubics; a triple infinity of rational space quartics; a fourfold infinity of elliptic space quartics. The general variety of order four in four dimensions contains a finite number of sets of four coplanar lines, and a finite number of sets of four non-intersecting lines, each latter set contained in a linear three-space. The number of these sets is found.

27. Dr. Oscar Zariski: *The existence problem for algebraic functions of two variables.*

The determination of the algebraic functions of two complex variables x, y, which possess a given branch curve D (a two-dimensional Riemann surface) in the real space S_4 of the variables x and y, depends upon the determination of the fundamental group G of the residual space S_4-D. The following theorems are stated: (1) If D, of order n, is irreducible and possesses double nodal points only, then G is the cyclic group of order n. (2) If the order n of the irreducible curve D is <6, then G is always the cyclic group of order n. (3) The first example of non-cyclic fundamental group is furnished by sextics with 6 cusps on a conic. Here G possesses two generators g_1, g_2, the generating relations being $g_1g_2g_1=g_2g_1g_2$, $(g_1g_2g_1)^2=1$. (4) If D possesses double nodal points only, and breaks up into k curves of order $n_1, n_2, \cdots, n_k$, respectively, then G is an abelian group generated by k generators $g_1, g_2, \cdots, g_k$, subjected to one generating relation: $g_1^{\rho}=1$, where ρ is the h.c.d. of $n_1, n_2, \cdots, n_k$. The following questions are raised: (a) What is the minimum number of cusps which an irreducible curve of given order and genus must possess, in order that G should not be cyclic? (b) Does the fundamental group depend upon the number of the cusps only, or does it also depend upon the position of the cusps?

28. Professor Malcolm Foster: *Note on cyclic systems.*

It is shown in this paper that the four focal surfaces of a cyclic system whose planes envelope a non-developable surface are the envelopes of the two families of spheres which contain the circles of the system and have their centers at the focal points of the associated cyclic congruence, and that the focal planes of this cyclic congruence bisect the arcs between corresponding focal points. If the planes of the circles of the system envelope the unit sphere, the associated cyclic congruence is normal; and the circles under an inversion go over into a second family of circles whose axes also

constitute a normal congruence. The paper also considers various cyclic systems lying in the XZ-plane of the moving trihedral of the surface of reference.

29. Mr. H. L. Dorwart: *The discussion of a construction of the rational plane quintic.*

The rational plane quintic, R_2^5, has a perspective conic, C, and if 2 of the 6 nodes of the R_2^5 are in a quadratic involution, it has 15 involution contact conics. The pairs in the involution are given by the extra intersections of conics on the remaining 4 nodes. Two of these conics will touch R_2^5, and will have a line, L, in common with C. Since L is perspective with C, and since each conic on the 4 nodes cuts out 2 points on L which are in (1, 1) correspondence with the extra 2 intersections of the conic and R_2^5, this gives a method for constructing the curve; it is mentioned by Morley in *The contact conics of the plane quintic curve* (The Johns Hopkins Circular, 1912). If two of the four nodes are at I and J, the pencil of conics is a pencil of circles, and the construction is simplified. We show a number of quintics, obtained by taking the different conic sections for the perspective conic, and by varying the relative positions of conic, line, and nodes.

30. Professor C. A. Rupp: *A geometric interpretation of linear dependence in n-space.*

We consider the geometrical implications of the algebraic statement that the set of homogeneous Grassmann-Plücker coordinates of an S_k in S_n is a linear combination of the sets of coordinates of several S_k's in the same S_n. The author has previously shown that $n+1$ lines (or S_{n-2}'s) which are linearly dependent are in the position of Schläfli, but here gives a simpler proof. A set of Schläfli or linearly dependent spaces determine and lie upon a variety of order and dimension $n-1$; Segre discussed this variety synthetically, as does Wong in a recent paper. By using the concept of linear dependence as central in the study of this variety, the author has found a method of obtaining its equation, and displays several examples in spaces of low dimensions. Another of Segre's synthetic discoveries was that of systems of "associated" lines and planes in S_4 and S_5. They are examples of a more general type of configuration, consisting of linearly dependent flat spaces; several of these configurations are discussed, and in particular it is shown how many spaces of such a set may be chosen arbitrarily.

31. Professor Nathan Altshiller-Court: *On nodal cubics circumscribed about a triangle.* First paper.

Consider the pencil of nodal cubics Σ circumscribed about the triangle $A_1A_2A_3$ and having in common the double point O and the two tangents at this point (see this Bulletin, vol. 26, pp. 203 and 114). Any point $P_{1,n}$ of the side $a_1 \equiv A_2A_3$ of the triangle $A_1A_2A_3$ determines a cubic Σ_n of the pencil Σ. (1) As the point $P_{1,n}$ describes the line a_1 the tangents to the respective cubics envelope a tricuspidal quartic (Δ_1) tangent to a_1 in A_2 and A_3, and to the line OA_1 in the points A_1. (2) The tangents drawn from $P_{1,n}$ to the respective cubics pass through two fixed points, namely the

points of intersection of the line OA_1 with (Δ_1). (3) The points of contact of these tangents lie on two fixed straight lines passing through O. (4) The line joining the points of contact of these two tangents passes through a fixed point. (5) The locus of the tangential of $P_{1,n}$ is a quartic curve circumscribed about the triangle $A_1A_2A_3$ and having a triple point at O. (6) The locus of the corresponding point of $P_{1,n}$ is a conic passing through the points O, A_1, A_2, A_3. (7) The line joining $P_{1,n}$ to its corresponding point envelopes a tricuspidal quartic, the line a_1 being doubly tangent to the curve.

32. Professor Nathan Altshiller-Court: *The Longchamps sphere. A definition.*

The usual definition of the Longchamps circle of a triangle (American Mathematical Monthly, 1926, p. 368) may be modified as follows (Mathesis, 1928, p. 156): Consider the three circles each having for center the vertex of a triangle and for the square of its radius the sum of the powers of the remaining two vertices with respect to the conjugate circle of the triangle. The orthogonal circle of these three circles is, by definition, the Longchamps circle of the triangle. This new form of the definition suggests an immediately obvious extension to space. It is shown that the sphere so defined has another characteristic property, analogous to one of the circle, namely, that it is the conjugate sphere of the anticomplementary tetrahedron of the given tetrahedron.

33. Professor A. D. Campbell: *Pencils of quadrics in the Galois fields of order* 2^n.

In the Galois fields of order 2^n the quadric $Q \equiv ax^2+by^2+cz^2+dw^2+fyz+gzx+hxy+lxw+myw+nzw=0$ has the discriminant $\Delta \equiv f^2l^2+g^2m^2+h^2n^2 \equiv (fl+gm+hn)^2$. Hence the discriminant Δ of a pencil of quadrics $P \equiv 2Q_1+\mu Q_2=0$ shows that P has one or two degenerate quadrics if $\Delta \not\equiv 0$, but if $\Delta \equiv 0$, P consists entirely of degenerate quadrics. In this paper are derived the classes of pencils of quadrics in the Galois fields of order 2^n and a typical pencil is given for each class. There occur pencils without degenerate quadrics. The most difficult but most interesting part of the problem is the discussion of the equivalence or nonequivalence of typical pencils.

34. Professor C. N. Reynolds: *On polyhedra.*

Given a finite set of integers, $a_n (n \geqq 3)$, when is it possible to construct a convex simply connected closed polyhedron bounded by a_n n-sided polygons with three polygons meeting at each vertex? It is known that Euler's theorem implies the necessity of the condition (1): $\Sigma_3(n-6)a_n+12=0$. This equation has 610 solutions for which $a \equiv \Sigma a_n < 11$, whereas only 140 of them are characteristic of known polyhedra. The most important propositions of the present paper yield a group of inequalities of the form (2): $\Sigma_3(n^2+pn+q)a_n \leqq f(a)$, which eliminate 347 of the 470 "extraneous solutions" of (1) mentioned above. As a increases beyond ten, the number of solutions of (1) eliminated by (2) increases numerically, but the

ratio of this number to the total number of extraneous solutions of (1) decreases.

35. Professor C. N. Reynolds: *On the problem of coloring maps in four colors.* Third paper.

Having considered (Annals of Mathematics, (2), vol. 28, pp. 1, 447) the geometry of maps which are irreducible with respect to reductions dealing with the immediate neighborhoods of regions, the author in the present paper deals with the gross structure of irreducible maps. The results of the preceding paper on polyhedra are found to be of value in describing such maps.

36. Miss Deborah M. Hickey: *Isometric circles on the sphere.*

Let the surface of the sphere be transformed into itself in a one-to-one and directly conformal manner. In this paper it is proved that, except for a rotation of the sphere, the areas of magnification and of diminution constitute respectively the smaller and larger spherical caps formed by a small circle on the sphere. Certain of the properties of this isometric circle are derived.

37. Dr. Jesse Douglas (National Research Fellow): *On the problem of Lagrange in the calculus of variations.*

The simplest form of this problem is to minimize the integral $\int F(x, y, z, y', z')dx$ under the auxiliary condition $\phi(x, y, z, y', z')=0$. A singular or so-called *anormal* case arises of which various interpretations have been given by Mayer (Mathematische Annalen, vol. 26 (1886)), von Escherich (Wiener Sitzungsberichte, vols. 107 (1898), 108 (1899), 110 (1901)), and Hahn (Mathematische Annalen, vol. 58 (1903–04)). The principal result of the present paper is a new interpretation of the anormal case to the effect that the characteristics of the auxiliary Monge equation are then extremals. Various other aspects of the problem are considered.

38. Dr. Jesse Douglas: *The affine, voluminar, and descriptive equivalence of spaces of K-spreads.*

(See the author's abstract *The geometry of systems of K-spreads*, this Bulletin, Jan.-Feb., 1928.) Two spaces of K-spreads are (1) affinely equivalent if they can be converted into one another by an arbitrary analytic transformation of coordinates, combined with a linear transformation of parameters, (2) voluminarly equivalent if they can be converted into one another by an arbitrary analytic transformation of coordinates combined with a transformation of parameters with constant jacobian, (3) descriptively equivalent if they can be converted into one another by an arbitrary analytic transformation of coordinates combined with an arbitrary analytic transformation of parameters. Necessary and sufficient conditions are established for the three types of equivalence. In particular, conditions are derived for affine, voluminar, and descriptive flatness, that is, for the

respective types of equivalence to the space $\partial^2 x^i/\partial u^\alpha \partial u^\beta = 0$. The methods and results are in the order of ideas of Christoffel's criteria for the equivalence of two quadratic differential forms, the criteria of Veblen and J. M. Thomas (Annals of Mathematics, Dec., 1925) for the projective (descriptive) equivalence of two restricted spaces of paths, and the author's criteria (abstract in this Bulletin, July-Aug., 1928) for the affine and descriptive equivalence of two general spaces of paths.

39. Dr. Jesse Douglas: *The Weyl tensor in the general geometry of paths.*

In the author's paper *The general geometry of paths* (Annals of Mathematics, (2), vol. 29, No. 2) a formula (7.11) is given for the Weyl tensor, analogous to the one given by J. M. Thomas (Proceedings of the National Academy of Sciences, vol. 11, pp. 207–209) for the restricted geometry of paths. In the present note a second formula is established for the Weyl tensor like the one given for the restricted theory by Veblen and T. Y. Thomas (Transactions of this Society, vol. 25, p. 560), being based on considering the effect on the curvature tensor of an arbitrary change of parameter on the paths.

40. Dr. Morris Marden (National Research Fellow): *On the zeros of linear partial fractions.*

This paper determines the locus of the zeros of the partial fraction $F=\sum_1^p m_i/(z-z_i)$, when each z_i varies over a given circular region C_i. Its results are applied to the location of the zeros of the derivative of a rational function, especially of a polynomial, to the location of the zeros of the Jacobian of two binary forms, and to a certain cross-ratio problem. Its theorems are essentially generalizations obtained analytically of theorems previously obtained by Lucas, Bôcher, and Walsh.

41. Professor W. W. Elliott: *Note on generalized Green's functions.*

In a previous paper (American Journal of Mathematics, vol. 50, p. 243) the existence of generalized Green's functions for compatible differential systems has been demonstrated, and the characteristic properties of certain types of symmetrical generalized Green's functions have been studied. In the present paper other types of symmetrical generalized Green's functions are discussed.

42. Dr. W. J. Trjitzinsky: *On quasi-analytic functions.*

Borel's monogenic functions, whose discovery antedates that of quasi-analytic functions of a real variable (Denjoy), are quasi-analytic in a complex domain C from which a point set of measure zero has been excluded. In this paper it is shown that Borel's functions belong to the first Denjoy class with certain restrictions on the domain of existence. Further, series of the form $\sum c_n/(x-a_n)$, which are quasi-analytic when the c_n satisfy conditions given by Borel and Carleman, are considered. With less re-

strictive specifications of the c_n, conditions are found under which such series belong to the first Denjoy class. Carleman's representation when initial values are given is extended. Necessary and sufficient conditions are found securing convergence of a sequence of polynomials to a quasi-analytic function. A new representation of a class of functions is given.

43. Professor R. L. Jeffery: *The uniform approximation by means of Riemann sums of a function defined by a Lebesgue integral.*

Let $f(x, y)$ defined on a bounded domain be summable in x for each y. The chief concern of this paper is to determine conditions on $f(x, y)$ under which $\int_a^b f(x, y)dx$ can be approximated by a Riemann sum $\sum f(\xi_i, y)(x_i - x_{i-1})$ uniformly in y. Extending a result due to Lebesgue, concerning the simultaneous approximation of p functions $g_1(x), \cdots, g_p(x)$ (Annales de Toulouse, (3), vol. 1, p. 33), the author shows that if $f(x, y)$ is bounded and continuous in y for each x, then for any subdivision with norm sufficiently small, there exists ξ_i on (x_{i-1}, x_i) such that $\left|\int_a^b f(x, y)dx - \sum f(\xi_i, y)(x_i - x_{i-1})\right| < \epsilon$, independent of y. Lebesgue remarks that his result holds for an infinite sequence $g_1(x), g_2(x), \cdots$. It seems that this statement needs some qualification, even when $g_n(x)$ is bounded in x and n.

44. Dr. Hillel Poritsky (National Research Fellow): *The billiard-ball problem on a convex table.*

The problem considered is the motion of a point in the plane inside a simple closed convex curve in a straight line, with reflection at the boundary so that the angle of incidence is equal to the angle of reflection. This problem, because of its lack of formal complication, is well suited for illustrating various methods in dynamics, and has been used by Birkhoff in connection with Poincaré's geometric theorem and his own extension of it (Acta Mathematica, vol. 50). The results obtained in the present paper are based largely on the existence of an invariant integral for the transformation associated with the reflection of paths at the boundary. Thus, the existence and distribution of certain periodic orbits as well as of orbits passing through two points or satisfying other conditions is inferred directly from the existence of the invariant integral, by simple geometric methods. The integrability of the problem is studied from various points of view leading to results of interest in projective and differential geometry. Morse's theory of extremals is found to apply, in spite of the discontinuity in the calculus of variations problem whose extremals constitute the above orbits.

45. Dr. Hillel Poritsky: *Cauchy-Green expansions.*

Under this title are considered certain expansions of analytic functions over points P inside a surface S, that express u as a sum of integrals over S involving the boundary values of u, $\nabla^2 u$, $\nabla^4 u, \cdots$, multiplied by certain simple functions of the distance from P and its normal derivative. These expansions form the analogue, for analytic functions of several variables, of the Cauchy integral for a function of a complex variable, and, more

precisely, of the Green "double-layer" integral for harmonic functions. A finite series of this type with a remainder is obtained by successive application of Green's theorem. It is then shown that if the number of terms be allowed to become infinite the remainder, under proper restrictions on S, will approach zero. The restrictions on S may be materially lightened if u satisfies certain partial differential equations.

46. Dr. Hillel Poritsky: *On certain integrals over spheres.*

It is shown that the linear operation on a function u that consists in replacing u over each of a family of concentric spherical surfaces by the function of the radius $\int udS/\int dS$ or the "average" of u over the spherical surface is permutable with the Laplacian operator. This property is further extended to certain other operators that may be described as leading from u to the various terms in the expansion of u in a series of spherical surface harmonics. From these simple facts are deduced a great many results, some familiar, some new, in the theory of harmonic and related functions. Among these are various laws of the spherical mean, integral representations for spherical harmonics and Bessel functions, and the reduction of certain expansions to Fourier expansions. The motif of the paper is geometric throughout.

47. Dr. Hillel Poritsky: *On certain oscillation theorems.*

It is shown that a homogeneous harmonic polynomial of degree n that does not vanish along a certain ray, l, through the origin, must change sign within the conical region about l in which the harmonic polynomial $p_n = r^n P_n(\cos\theta)$ stays positive, where r is the distance from the origin and θ is the angle that rays through the origin form with l. Analogous results are established for bi-harmonic, tri-harmonic, $\cdots$, polynomials, as well as for solutions of certain partial differential equations. In this connection a necessary and sufficient condition is derived and utilized for the existence of a linear combination of several given functions of a single variable that does not vanish throughout a certain interval. It is then shown how some of the former results may be extended to regions of arbitrary shape by the use of the characteristic parameter values associated with these regions.

48. Professor E. R. Hedrick: *Integrals along given curves in the complex domain.*

In this paper the author discusses integrals taken along given curves in the complex domain, for functions that are non-analytic. It is shown that certain types of theorems are easily extended from the traditional theory to include integrals of non-analytic functions whenever the paths are fixed in advance. Thus the theorems recently stated by Pfeiffer (this Bulletin, vol. 34 (1928), pp. 656–664) can be thus extended.

49. Professor Marston Morse: *A generalization of Sturm's separation theorem in n dimensions.*

The author considers a set of n self-adjoint, second-order, linear, homogeneous differential equations in n dependent variables (y) and an in-

dependent variable x. The matrix of the coefficients of the second derivatives is assumed to have a characteristic determinant all of whose roots are positive. Any such system of differential equations are shown to be the Jacobi differential equations of an easily determined integral in the calculus of variations. With this point of view, the author considers the determinants $D(x)$ of systems of n linearly independent solutions which are mutually conjugate in the sense of von Escherich. He shows that if one of these determinants vanishes $n+r$ times on a closed interval of the x axis, any other of these determinants will vanish at least r times on the same closed interval. This theorem and others appeared as corollaries of a fundamental theorem on the type number of a quadratic form discovered and used in a development of the calculus of variations in the large.

50. Professor J. L. Walsh: *On approximation by rational functions to an arbitrary function of a complex variable.*

In the plane of the complex variable z, let M be a closed point set whose boundary consists of a finite number of Jordan arcs which do not separate the plane into more than a finite number of regions. Then an arbitrary function $f(z)$, continuous on M and analytic in the interior points of M can be uniformly approximated on M as closely as desired by a rational function of z.

51. Professor Orrin Frink: *Functions defined over an abstract set.*

In this paper the theorems that a function continuous over a connected set assumes all values between its maximum and minimum, and that a function continuous over a compact-in-itself set is bounded and assumes its upper and lower bounds, the Heine-Borel theorem, and other standard theorems of real function theory, are proved either for the general topological space or for a space subject to one weak postulate. The theorems are more general than the corresponding ones proved by Fréchet for neighborhood spaces.

52. Professor James McGiffert: *Two closed forms of particular solutions of the linear differential equation of second order* $(A+Bx^2)y''+Cxy'+Dy=0$.

We assume a solution in series form, $y=\sum_{m=0}^{\infty}a_m x^m$. Substituting this value for y, and its two derivatives for y' and y'', and equating the combined coefficients of x^m to zero, we obtain the recurrence relation $a_{m+2}=-(B/A)[(m^2+(C-B)m/B+D/B)/((m+1)(m+2))]a_m$. This rather complicated recurrence relation does not in general lead to useful forms of particular solutions. We may, however, so choose the coefficients, that the numerator of the fraction in brackets shall have $m+1$ or $m+2$ as a factor. Then the recurrence relation will be much simpler, and will reduce to two binomial forms, depending on the value of D in terms of B and C, as follows: (1) when $D=C-2B$, (2) when $D=2C-6B$. In the first case we obtain the solution $y=(A+Bx^2)^{1-c/(2B)}$, and in the second case, $y=x(A+Bx^2)^{1-c/(2B)}$.

Legendre's equation is obtained by making $A=1$, $B=-1$, $C=-2$, and $D=k(k+1)$.

53. Professor W. M. Whyburn: *Non-isolated critical points of functions.*

A function $f(x_1, \cdots, x_n)$ is studied, where the first partial derivatives of f are assumed continuous in a closed and connected portion R of the space of the n real variables $x_1, \cdots, x_n$. Critical points of f are considered under hypotheses which are not sufficient to cause these points to be isolated, and the notion of "critical sets" is developed. (For the case of isolated critical points, see Marston Morse, Transactions of this Society, vol. 27 (1925), pp. 345–396.) In particular, it is shown that if f has two distinct "minimal sets" in R it must have at least one other critical point in R. Upon specialization, this result yields Bieberbach's form of Birkhoff's "minimax principle" (see Bieberbach, *Differentialgleichungen*, Berlin, 1927, p. 140).

54. Professor Philip Franklin: *Equimomental systems.*

In this note we investigate certain simple configurations of mass points which are equimomental to any given rigid system, that is, have the same moment of inertia about all axes in space. It is shown that any system is equimomental to four mass points, which may be selected in a triple infinity of ways. The ratio of the masses at the points may be arbitrarily assigned, and in general uniquely determines the equivalent set. The tetrahedron with vertices at the four points has its opposite edges conjugate with respect to the homogeneous solid ellipsoid equimomental to the system. Some additional geometrical relations are given.

55. Mr. C. F. Muckenhoupt: *Vibrating systems and almost periodic functions.*

A vibrating system such as a vibrating string or membrane has its energy invariant under proper assumptions as to boundary conditions. By means of the finiteness of this invariant, together with similar invariants of higher order, it is proved that a vibrating membrane or string returns arbitrarily close to its initial position within some finite time. It follows from this with another definition of distance that the membrane or string not merely returns to the neighborhood of its initial position but that the difference between the displacement of the membrane or string and its original displacement has an energy which is arbitrarily small. From this the almost periodic character of the vibration may be proved. It is possible on this basis to discuss the closure of the set of characteristic functions of a vibration problem.

56. Professor Yusuke Hagihara: *Complete stability of a dynamical system with long-period variations.*

The author deals with a dynamical system with long-period variations superposed on short-period variations, from the stand-point of complete

stability as defined by Birkhoff. Then a certain number of parameters are introduced in the differential equations of the dynamical system, and the results compared with Poincaré's work in the problem of three bodies.

57. Professor C. A. Garabedian: *Concerning two-dimensional solutions of the problem of the thick circular plate loaded at its center.*

The author's corrections of de Saint-Venant's solutions (Journal de l'Ecole Polytechnique, (2), vol. 26 (1927)) followed the notation of de Saint-Venant and employed only devices found in the "Note du paragraphe 45." The present paper complements the foregoing in that it examines critically contributions subsequent to de Saint-Venant. Only by thus completing the study of the problem does it seem possible effectively to remove the obscurities which have pervaded the two-dimensional solutions. By including also some consideration of recently developed three-dimensional solutions, there is secured, on this whole question of the thick circular plate centrally loaded, a perspective hitherto lacking.

58. Professor E. G. Keller: *Harmonic functions for an anchor ring of elliptical meridian section.* Preliminary communication.

In this paper a solution of Laplace's equation in series in elliptical toroidal coordinates is obtained. Circular symmetry is assumed. Each term of the series is a particular solution of the equation. Successive changes of independent variables and the same change of dependent variable as that employed by C. Neumann with reference to a circular tore (*Theorie der Elektricitäts- und Wärme-Vertheilung in einem Ringe*) yield the equation $\xi^2 U_{\xi\xi}+2\xi\eta U_{\xi\eta}+\eta^2 U_{\eta\eta}+U_{\zeta\zeta}+\frac{1}{4}U=0$. A solution of this equation is $T_n(r)\cdot Y_n(\zeta)$, where T_n and Y_n are solutions of ordinary differential equations. The latter may be written as a series of Bessel coefficients of increasing order. The function T_1 is reducible to the Bessel function of order zero and of purely imaginary argument. Recurrence formulas for T_n are obtainable. If an expansion of a function $f(r)$, in either of the series $\sum_{n=0}^{\infty} A_n T_n$ or $\sum_{m=0}^{\infty} B_m T_n(a_m r)$, exists, the coefficients are readily expressed as definite integrals.

59. Dr. L. T. E. Thompson: *Note on empirical summaries in engineering ballistics.*

It is essential to facilitate extension (in scale) of experimental results bearing on certain ballistic problems. The study of wear or erosion phenomena, an aspect of the general uniformity problem of the gun cycle, may be developed by means of the ratio $f(C/C_0,\ l/d)\equiv P(C/C_0)/Z(l/d) = E/(kd^n)$; rate of bore enlargement E, powder charge C (datum value C_0), length and diameter of bore l and d; k and n are constants. Introducing appropriate assumptions with respect to maximum permissible enlargement, a relative life-of-gun formula is obtained. For the representation of plate punching data, it is practicable to identify a numerical (penetration)

coefficient $F(e/d, \theta) = K_1 M^{1/2} V \cos \theta / e^{1/2} d$; projectile mass M, diameter d, limit velocity V, plate thickness e, obliquity of impact θ (see the author's paper in Mémorial de l'Artillérie Française, vol. 6 (1927), p. 1255, note 3); by subsequent test this form is justified, especially in case application to moderate departure from similarity of projectiles is contemplated. It is required to determine one set of experimental F curves for each class of projectiles and of plate characteristics. A laminated or divided armor coefficient can be identified by means of these and a corresponding absorption summary.

60. Professor Norbert Wiener: *Hermitian polynomials and Fourier analysis.*

The nth Hermitian function differs from its Fourier transform by the factor i^n. The present paper is devoted to the study of the transformation which multiplies the nth Hermitian function by ω^n, where ω lies on the unit circle.

61. Mr. A. C. Berry: *Spectral intensity of an n-dimensional function.*

A proof of the complex n-dimensional form of the fundamental theorem in the Plancherel theory of the Fourier transform is given in the first part of the present paper. Although it has seemed necessary to use cartesian coordinates in giving the proof, the theorem itself is vectorial. The remainder of the paper discusses the intensity of the spectrum of a complex point function of a point in real n-dimensional space. A sufficient condition that the total spectral intensity be finite is given. This discussion is the vectorial, n-dimensional generalization of the first half of a recent article by N. Wiener (*The spectrum of an arbitrary function*, Proceedings of the London Mathematical Society, (2), vol. 27, No. 7).

62. Mr. S. B. Littauer: *On a theorem of M. Jacob.*

The author proves a theorem of Jacob concerning the equivalents of two forms of mean of a function by reducing this theorem to a particular case of a type of Tauberian theorem treated by Wiener. The reduction in one direction is immediate; in the other, a slight extension of Wiener's methods is necessary.

63. Professor O. D. Kellogg and Dr. F. Vasilesco: *A contribution to the study of a certain lemma in potential theory.*

The lemma is as follows: every closed bounded set of points of positive capacity contains a regular point. It has been proved in the case of the logarithmic potential, but in space it is still in doubt. In order to make progress with the problem, it is necessary to have further knowledge of the properties of regular and exceptional points. In the present paper, the authors derive several new forms of criteria for regular and exceptional points. They show that the capacity of the portion of a set E in a sphere of fixed radius about P is a function of P which is upper semicontinuous,

and that a certain modified form of the series of Wiener is a function of P which is of the second class, in the sense of Baire. In certain cases this function may reduce to one of the first class.

64. Professor H. S. Vandiver: *On Fermat's last theorem.*

In various papers published during the last fifteen years the writer has obtained a number of results concerning Fermat's last theorem. In the present paper all these results are obtained from two general methods of approaching the problem. Some new criteria are also derived by the use of these methods. When applied to special exponents they yield the result that if $n>2$, $x^n+y^n+z^n=0$ is impossible in rational integers x, y, and z, none zero, for every n less than 211.

65. Professors H. S. Vandiver and Elizabeth T. Stafford: *Determination of some properly irregular cyclotomic fields.*

The integer which expresses the number of classes of ideals in a cyclotomic field defined by $e^{2\pi i/l}$, l an odd prime, is generally written as the product of two factors which are also rational integers, called the first and the second factor of the class number respectively, that is, $h=h_1h_2$. A necessary but not a sufficient condition that h_2 be divisible by l is that the first factor h_1 be divisible by l. In the present paper it is found by extensive computations that h_1 is divisible by l for l less than 211 only when $l=37$, 59, 67, 101, 103, 131, 149, 157, and that h_2 is not divisible by l in any of these cases. The paper includes proofs of the formulas employed, and descriptions of the many devices to shorten the computations.

66. Mr. H. T. Engstrom: *Theorems on common index divisors of algebraic fields.*

This paper gives the determination of common index divisors for algebraic fields of third, fourth, and fifth degrees. For these fields the common index divisor is defined by the prime ideal decomposition of the corresponding primes. For fields of ninth degree a special example is constructed showing that this is not the case in fields of ninth degree or higher.

67. Professor W. L. G. Williams: *On absolutely isobaric formal modular concomitants.*

The object of the present paper is to show that certain formal modular concomitants of binary forms are congruent to algebraic concomitants and to point out some applications of this fact in the theory of numbers. As a typical theorem the following may be cited: An absolutely isobaric formal modular invariant, mod p, a prime, of a binary form with binomial coefficients, no one of which is divisible by p, is congruent, mod p, to an algebraic invariant of this form. In this and other cases the author finds the algebraic concomitant to which a given formal modular concomitant is congruent. The following identical congruence will illustrate the applications to the theory of numbers. The coefficient of $x^{p-1}y^{p-1}$ in the expansion of $(ax^2+2bxy+cy^2)^k(x\xi-y\eta)^{2l}$ is congruent, mod p, to

$K(ac-b^2)^{k-(p-1)/2}(a\xi^2+2b\xi\eta+c\eta^2)^l$, where k, K, and l have the values $(p-1)/2 \leqq k \leqq p-1$, $K=(-1)^{(p-1)/2}2^{2k}\binom{2l}{l}$, and $l=p-1-k$.

68. Professor Orrin Frink: *The structure of finite algebras.*

This paper deals with algebras with a finite number of elements, the addition group being abelian, but multiplication not necessarily commutative. Such algebras are important in the number theory of linear algebras. It is shown that such an algebra is reducible unless of prime power order. Most of the standard theorems of linear algebras go through, since their proofs do not depend on the existence of a scalar field, but on idempotents and the general theory of complexes. A special study is made of the important case of associative algebras of prime power order, which can be represented as algebras of matrices whose elements are residue classes of integers modulo a prime power.

69. Professor Orrin Frink: *A theorem in boolean algebra.*

The theorem states that the result of substituting expressions which are in the completely expanded form in a function of several variables is given by multiplying the expressions together symbolically, preserving order, and interpreting any symbolic product which occurs as a coefficient in the result as meaning the result of substituting the factors of this product in the given function in the order in which they occur. It is shown how the theorem is used to simplify computation.

70. Dr. John Williamson: *The complete system of two quadratic quaternary forms.*

This paper obtains by a new method the system determined by H. W. Turnbull (Proceedings of the London Mathematical Society, (2), vol. 18 (1919)). Starting from the complete system, already known (Turnbull and Williamson, Proceedings of the Royal Society of Edinburgh, vol. 45 (1925)), which involves three cogredient sets of point coordinates x, y, and z, it is shown how the variables x, y, and z may be convolved to give a complete system of concomitants expressed in terms of point, line, and plane coordinates; that is, in terms of the variables x, $p=(xy)$, and $u=(xyz)$. While the results obtained are not new, the method has the advantage that it can be extended to finding, if not the complete system, at least a prepared system, for two quadratics in five or six variables. Incidentally, the number of irreducible concomitants obtained by Turnbull is diminished from 125 to 122.

71. Dr. A. A. Albert (National Research Fellow): *On the structure of normal division algebras.*

It is first shown that in a division algebra if y satisfies the minimum equation of x then y is a transform of x. The grade of an element is defined as the degree of its minimum equation. The type of an equation is defined as a function of the number of its ordinary complex roots rationally expressible in terms of one of them. The type of an element and that of an algebra are defined, and the structure of algebras of type S_k is considered.

The concept of "known algebras" is broadened and a necessary and sufficient condition that an algebra be a known algebra over a given subalgebra is found. It is demonstrated that if the direct product of two normal division algebras, each of the maximum type, is a division algebra, then it is a normal division algebra of the maximum type.

72. Dr. A. A. Albert: *On the group of the rank function of a normal division algebra.*

A new theorem in the Hilbert theory of irreducible polynomials whose coefficients are rational functions with coefficients in a field K of several parameters is proved in this paper. The theorem states that, for any complex values of the parameters, the group of the resulting equation with respect to the field obtained by adjunction to K of the resulting coefficients is a sub-group of the group of the original polynomial. This theorem is applied to the rank function of any normal division algebra, and it is shown that this function has the symmetric group. As a consequence of this it is seen that every normal division algebra in n^2 units contains an element whose minimum equation has degree n and the symmetric group.

73. Dr. A. A. Albert: *Normal division algebras in* $4p^2$ *units, p an odd prime.*

Normal division algebras in $4p^2$ units, p an odd prime, are considered. Algebras of type R_3 are shown to be known algebras in the sense previously defined. A study is made of conditions that an algebra A in $4p^2$ units contain an element x of grade $2p$ whose minimum equation is $\phi(\omega^2)=0$. It is shown that A is a known algebra over the Σ algebra obtained from x if and only if the equation $\varsigma(\rho)=0$ is cyclic. This is the case considered by L. E. Dickson in his note in the September, 1928, number of this Bulletin. It is proved that these algebras are of type R_{2p} and hence not new.

74. Dr. A. A. Albert: *Normal division algebras in* 36 *units.*

In the present paper all normal division algebras in 36 units of type R_3 are shown by an elaborate argument to be of type R_3. The algebras of type R_3 were shown, in another paper, to be known algebras. A necessary and sufficient condition that the new algebras thus found be actually not equivalent to algebras previously constructed is given. Since the assumption of type R_2 is the weakest assumption of this kind that it is possible to make, the discussion is quite general.

75. Professor Oystein Ore: *An explicit case of "Zerlegungs"-groups.*

Several of the general properties of the "Verzweigungs"-groups introduced by Hilbert are known, but except in the case of abelian fields, their structure is unknown. The author has shown in a previous paper that the prime ideals of a Galois field can be divided into two types, binomial and trinomial. For the binomial types the complete structure of the corresponding non-abelian Hilbert group is determined.

76. Professor H. W. Brinkmann: *On the group characteristics of multiply transitive permutation groups.*

A permutation group on n letters is, of course, a sub-group of the symmetric group on those letters. An irreducible homogeneous linear group isomorphic to the symmetric group thus furnishes a homogeneous linear group isomorphic to the given group. But this group is usually reducible. The present paper investigates the irreducible constituents of these groups. For example, it is shown that for a t-fold transitive group certain of these groups are irreducible. The irreducible ones at once give characteristics of the group in question.

77. Professor H. W. Brinkmann: *Imaginary quadratic fields in which each genus contains but one class.* Second paper.

In a previous paper (read at the summer meeting of the Society, 1928) fields containing two genera, each of which contains one class, were discussed. The case of fields with four such genera is taken up in the present paper. The conditions that a field be of this type are derived and applications are made to elementary number theory.

78. Professor H. W. Brinkmann: *Note on algebraic equations not solvable by radicals.*

It is well known that irreducible equations (with rational coefficients) exist for every degree which cannot be solved by radicals. It is the purpose of this note to give explicitly examples of such equations for prime degree. Thus we find that $x^p-px+p=0$ is irreducible and cannot be solved by radicals for any prime $p\geqq 5$.

79. Professor Raymond Garver: *On the removal of four terms from an equation by means of a Tschirnhaus transformation.*

This paper has appeared in full in the January-February number of this Bulletin.

80. Professor Raymond Garver: *Concerning certain functions of Brioschi.*

A paper of Brioschi's (see his Opere, vol. 3, pp. 293–299) proves the existence, for an equation of odd degree n, of $(n-1)/2$ polynomial functions possessing certain properties which make them of some value in connection with Tschirnhaus transformations. The present paper gives a simpler proof of the theorem, and considers its extension to equations of even degree.

81. Mr. R. E. Powers: *Note on the Pellian equation.*

In 1877 Artemas Martin found that the smallest integral value of x which satisfies the Pellian equation $x^2-9817y^2=1$ contains 97 digits (L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 383). I have verified this result, and find that the corresponding value of y contains 95 digits.

82. Professor D. C. Harkin: *The abstract identity of modular systems and ideals.*

By the method of abstraction and transformation by formal equivalence, Dedekind's theory of ideals (1871 and later) and Kronecker's theory of modular systems, as developed in his famous *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* of 1882, are shown to be abstractly identical. A logically consistent theory of forms (or ideals) is built up, leading finally to the unique decomposition of forms into prime forms. The terminology and external form of the general theory developed agrees closely with Kronecker, but the root idea and fundamental meaning is taken almost bodily from Dedekind. Various possible interpretations of the abstract general theory are indicated: the theory of algebraic forms, with either the Gaussian transformation or the Kummer composition of forms as fundamental; geometrically, the correspondence of an ideal to a system of points on a fixed fundamental curve, a principal ideal to the complete intersection system of a fundamental curve with another algebraic curve, with equivalence translated into coresiduality.

83. Professor R. L. Moore: *On the structure of a continuum.*

It is shown that if M is a bounded plane continuum and no upper semi-continuous collection of mutually exclusive subcontinua of M is homeomorphic with a domain, then there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that G is a dendron with respect to its elements and such that if H is another collection satisfying these conditions then every element of H is made up of elements of G. The elements of G will be called *dendratomic* subsets of M. For each non cut point P of the bounded plane continuum M let M_P denote the set of all points X of M such that X is not separated from P in M by any one point. For each cut point P of M let M_P denote P. If M is a continuous curve every M_P is a cyclic element of M in the sense of G. T. Whyburn and if Q is an upper semi-continuous collection of mutually exclusive continua filling up M and no M_P contains points of two different continua of Q then every continuum of Q is made up of continua of the collection G_1 defined on page 700 of volume 34 of this Bulletin.

84. Professor R. G. Lubben: *The metrization problem and complete spaces.*

The author shows that certain well known conditions for the metrization of a topological space may be strengthened so as to give necessary and sufficient conditions that such a space be complete. It follows from this result that a locally compact, metric space is complete. A *universal complete enclosure* of a metric space M is by definition a complete space N which contains M and admits a metric d such that if T is any complete space whatever containing M, and d' is a metric with reference to which T is complete, then if a sequence of points in M is a fundamental sequence with reference to d' it is a fundamental sequence with reference to d. In order that a metric space admit a universal complete enclosure, it is necessary

and sufficient that it be separable; in order that it admit a minimal universal complete enclosure, it is necessary and sufficient that it be locally compact and separable.

85. Professor R. G. Lubben: *Concerning the distributive property.*

The author establishes the following results. (1) If a space L has the distributive property (see Transactions of this Society, vol. 30 (1928), p. 669), then every point set in it having the power of the continuum has a limit point. (2) In a space S having the distributive property, every point set having the power of the continuum *contains* a limit point of itself. (3) If a space S has the distributive property, then every regular, locally compact sub-space of it has this property. (4) Every space S which has the distributive property and satisfies the first countability axiom is regular and locally compact. (5) A space S which is regular, is locally compact, and has the Lindelöf property, has the distributive property.

86. Mr. J. H. Roberts: *On a problem of C. Kuratowski's concerning upper semi-continuous collections.*

In a letter to R. L. Moore dated April 9, 1927, C. Kuratowski raised the question as to whether or not there exists an upper semi-continuous collection X of mutually exclusive continua no one of which is a point such that (1) the sum of the continua of the collection X fills a square plus its interior, and (2) if each continuum of the collection X is regarded as a point, the space so obtained is in continuous one-to-one correspondence with a square plus its interior. In the present paper this question is answered in the affirmative.

87. Mr. J. H. Roberts: *Upper semi-continuous collections of continua some of which are unbounded.*

R. L. Moore has proved that if, in a plane S, G is an upper semi-continuous collection of mutually exclusive *bounded* continua no one of which separates S and such that every point of S belongs to some continuum of G, then, if the continua of G are regarded as points, the space so obtained is in continuous one-to-one correspondence with the plane S. In the present paper it is shown that if G is an upper semi-continuous collection of mutually exclusive continua such that (1) every point of the plane S belongs to some continuum of G, (2) at least two continua of G are unbounded, and (3) the space of elements of G is metric, then the space of elements of G is in continuous one-to-one correspondence with a subset of the plane S.

88. Dr. W. L. Ayres (National Research Fellow): *Continuous curves in which every arc may be extended.*

An arc α of a set K is said to be extendible in both directions in K if K contains an arc β such that every point of α is an interior point of β. It is shown in this paper that a plane continuous curve M has the property that every arc is extendible in both directions in M if and only if one of the two following conditions holds: (a) no points of M may be ϵ-separated by an acyclic continuous curve; (b) the sum B of the boundaries of the

complementary domains of M is closed and every point of B is either a cut point of M or belongs to the boundaries of at least two complementary domains of M. If either (a) or (b) is true, then every point of M is a point of B. For any point P of M and any positive number ϵ, there exists a simple closed curve J of diameter less than ϵ and enclosing P and such that the set $M \cdot J$ consists of a finite number of components.

89. Dr. W. L. Ayres: *On generalizations of continuous curves.*

If $x_i (i = 1, 2, \cdots, n)$, is a continuous function of t, the set of all points $(x_1, x_2, \cdots, x_n)$ for $0 \leqq t \leqq 1$ is called a continuous curve. Two possible generalizations of this notion may be had by allowing t to take on all values or all values $t \geqq 0$. In this note it is shown that these two generalizations are equivalent and the type is characterized.

90. Professor G. T. Whyburn: *Cut points of connected sets and of continua.*

Let M be any continuum (in a locally compact, metric, and separable space), let A and B be any two mutually exclusive closed subsets of M, let K be the set of all those points of M which separate A and B in M, and let G denote the set of all cut points of M. In this paper it is shown that (1) $K+A+B$ contains every point of $\overline{K}$ which is a Menger regular point of $M+P$; (2) K is the sum of a G_δ set and a countable set; (3) G is a $G_{\delta\sigma}$, that is, the sum of a countable number of G_δ sets; (4) if K is uncountable, it contains a perfect set; (6) if M is compact and irreducible between A and B, then K contains every point of $M-(A+B)$ at which M is connected im kleinen. Result (1) is true for any *connected set* M. Some other theorems of a similar character are also obtained.

91. Professor G. T. Whyburn: *On the structure of connected and connected im kleinen point sets.*

It is shown in this paper that all connected and connected im kleinen point sets M in a locally compact, metric, and separable space admit of decomposition into elements highly analogous to the cyclic elements of a continuous curve. Elements (a) of such a set M are connected subsets of M which have no cut point and which are saturated with respect to this property; elements (b) of M are cut points of M; and elements (c) of M, called end points, are points P of M such that a sequence V of points of M exists converging to P and such that each point of M distinct from P is separated in M from P by all save a finite number of points of V. These correspond respectively to the maximal cyclic curves, cut points, and end points of a continuous curve; and it is shown that the structure of any such set M with respect to its elements (a), (b), and (c) is very similar to the structure of a continuous curve with respect to its cyclic elements (see American Journal of Mathematics, vol. 50, pp. 167–194). One interesting theorem proved is that every connected subset of the set of all elements of classes (b) and (c) is arcwise connected.

92. Professor G. T. Whyburn: *Continuous curves and arc sums.*

Menger has suggested the problem of characterizing a continuous curve which is the sum of a countable number of simple continuous arcs. In this connection the following theorems are proved. (1) The continuous curve M is the sum of a countable number of arcs if and only if (a) the end points of M are countable, and (b) each maximal cyclic curve of M is the sum of a countable number of arcs. (2) The boundary N of a complementary domain of a plane continuous curve is the sum of a countable number of arcs if and only if the end points of N are countable. (3) The set K of all the local cut points of any continuous curve M is a subset of the sum of a countable number of arcs which lie in M; and if $M-K$ is countable, M is the sum of a countable number of arcs.

93. Professor James Pierpont: *On the motion of a rigid body about a fixed point in space of constant curvature.*

The first to consider the motion of a solid in elliptic space was Clifford, in a paper read before the London Mathematical Society in 1874, but not published until 1882. This was followed by a paper by R. S. Heath published in 1884 in the Transactions of the Royal Society of London. Finally, de Francesco treated this problem in two papers in the Atti of the Academy of Science of Naples in 1900. Clifford employs biquaternions, Heath and de Francesco use the six homogeneous coordinates of a right line tied down by two biquadratic relations; de Francesco considers hyperbolic as well as elliptic space, and he also considers in detail special cases, which Heath does not do. The present author treats another special case, viz., the Maxwell top in space of constant curvature. In this important case he shows that homogeneous line coordinates are quite unnecessary, from which follows a great gain on the analytical side of the problem. With the same degree of accuracy as employed in classical mechanics we have the notion of center of mass. From this it results that the motion of the top in elliptic (hyperbolic) space would appear to an observer in euclidean space as that of a top with certain assigned moments of inertia and a certain center of mass.

94. Dr. A. J. Lotka: *Notes on Volterra's mathematical theory of biological associations.*

Volterra has set up a system of differential equations to represent the course of events in a system comprising two biological species competing for a common food supply. He has discussed certain properties of the functions defined by these equations, but has not offered any general solution. Such a solution in form of an exponential series is now given, the case discussed by Volterra being a special case of the general theory of the growth of n populations living together in a common environment, as developed in prior publications by the present author. The case discussed by Volterra may also be regarded as the Verhulst-Pearl law of growth for two competing populations. The independent variable t (time) can be

eliminated from the system of differential equations, which then contains only two variables. The *isoclines* of the integral curves of the system are investigated, and a map of the family of integral curves is prepared by their aid. The discussion of some other cases treated by Volterra is reserved for a future occasion. Application of the method of analysis to economic problems (for example, competition of two or more producers supplying the same market) is suggested.

95. Dr. P. R. Rider: *On the moments of the frequency distribution of third moments.*

This paper develops formulas for the moments of the frequency distribution of third moments of samples of n from an infinite population. The first and second moments are given for any type of population, the third and fourth moments for a normal population.

96. Dr. P. R. Rider: *The effect of skewness in the sampled population on the frequency distribution of the ratio of mean to standard deviation in small samples.* Preliminary communication.

The frequency distribution of the ratio of mean to standard deviation in samples of n from a normal population has been rather thoroughly discussed by "Student" and others. In samples from a normal population the mean and the square of the standard deviation are uncorrelated; for other types of population correlation between them exists. Shewhart and Winters have made an experimental study of the distribution of the ratio of mean to standard deviation in samples from certain non-normal populations, and have given their results in a recent number of the Journal of the American Statistical Association. The present paper endeavors to shed more light on the effect of skewness in the sampled population on the distribution of this ratio.

97. Professor A. H. Copeland: *A proof that almost every number is admissible and is associated with the probability one half.*

In this paper it is proved that in a certain sense almost every event satisfies the fundamental assumptions of the theory of probability, and moreover almost every event has the probability one-half. The sense in which this statement is true is as follows. If an event behaves in accordance with the assumptions of the theory of probability, then the history of successes and failures of this event must be represented by an admissible number. The admissible numbers all lie in the interval from zero to one, and the measure of the set of admissible numbers is equal to one. That is, almost every number in the unit interval is admissible. Furthermore it is proved that almost every admissible number is associated with the probability one-half.

98. Professor C. F. Roos: *A mathematical theory of the business cycle.*

Many economists have empirically investigated the changes in prices over a period of time, and have evolved an empirical theory generally known as the theory of the business cycle, which, in brief, states that prices are periodic functions of the time. Other equally competent investigators have branded this theory as a myth, and have exhibited empirical curves to support their contentions. The present paper gives a purely theoretical discussion of prices, and shows that these views, rather than being contradictory, are both only partial manifestations of an underlying theory. The hypothesis that a producer endeavors to determine his price so that he obtains a maximum profit over a period of time requires that the price satisfy a certain differential equation. This differential equation has ten distinct types of solutions, all of which seem to have economic significance. A truly periodic solution supporting the theory of the business cycle can result only if interest is neglected. Conditions are obtained that will lead to the ten different types of solutions; all the results are stated in a form that will allow experimental investigation.

99. Mr. H. H. Pixley: *A theory of discontinuous price and production curves.*

In this paper the author considers a generalization of Roos's problem of depreciation and replacement in the mathematical theory of economics, and removes the restrictions of continuity of price-time and rate of production-time curves at the time of replacement. Some applications are made of the general mathematical theory given in the author's earlier paper *Discontinuous solutions in the problem of depreciation and replacement*, in considering special problems believed to have actual economic significance. The present paper consists essentially of applications.

R. G. D. RICHARDSON,
Secretary

THE CHRISTMAS MEETING IN CHICAGO

The two hundred sixty-sixth, thirty-first regular Western, meeting of the Society was held at the University of Chicago on Monday, December 31, 1928. About sixty persons attended the meeting, among whom were the following fifty members of the Society:

R. W. Babcock, Bartky, Beenken, O. E. Brown, L. H. Bunyan, G. N. Carmichael, R. D. Carmichael, Chittenden, A. G. Clark, H. H. Conwell, C. F. Craig, Crathorne, Curtiss, Dickson, Escott, Everett, Feltges, Gouwens, L. M. Graves, Griffiths, V. G. Grove, Holgate, Ingraham, B. W. Jones, B. F. Kimball, E. P. Lane, Langer, LaPaz, Leavens, Logsdon, McCoy, March, Marshall, E. H. Moore, T. W. Moore, Pepper, Pettit, H. L. Rietz, Ross, Roth, Sanger, Schottenfels, Simmons, Wall, Warren Weaver, Weiss, F. E. Wood, Frederick Wood, Roscoe Woods, Worth.

In the evening thirty-seven members and guests attended dinner in the Del Prado Hotel. Professor Lane acted as toastmaster and, before the dinner, called upon Professor E. H. Moore, who could not remain for the dinner. Professor Moore spoke for a few minutes, welcoming the members and guests to Chicago. After the dinner the toastmaster called upon Professor R. D. Carmichael, who told of the New York meeting, Professor Chittenden, who talked of the coming Des Moines meeting, Professor Ingraham, who spoke of the problems of the location of meetings in the Mid-West and suggestions that had been made towards increasing the interest of the programs, and Miss Weiss, who spoke for the ladies.

The papers whose abstracts appear below were read as follows: papers 1–9 in the morning, and papers 10–19 in the afternoon. Papers 6, 8, 9 and 15–19 were read by title, the rest in person. Professor Marshall presided in the morning and Professor Curtiss in the afternoon.

1. Dr. B. W. Jones: *The number of representations by certain positive ternary quadratic forms.*

The number of solutions of $x^2+y^2+z^2=n$ in integers for any positive integer n is a known function of the class number of n. But the number of

integral solutions of $a\,x^2+by^2+cz^2=n$, where a, b and c are positive integers, is not, in general, known. In this paper, by means of a transformation taking the form $x^2+2y^2+3z^2$ into $x^2+y^2+z^2$, the number of representations of n by the former is shown to be a simple function of the number of representations by the latter of $6n$ and, in some cases, other multiples of n. From these results can be found the number of representations of n by the former in terms of the class numbers of certain multiples of n. This type of relationship has been considered only incidentally by Nazimoff and Uspensky and only fragmentary results have been obtained for this form. It is demonstrated that not only the form mentioned but many other new forms, seemingly unrelated to the sum of three squares, yield to similar treatment.

2. Dr. Lois W. Griffiths: *A generalization of the theorem of Fermat and Cauchy on polygonal numbers.*

This theorem, that every positive integer is the sum of $m+2$ polygonal numbers of order $m+2$ and that all but four of these may be taken to be 0 or 1, has been extensively generalized by Dickson. Another generalization is suggested by the following restatement of the theorem. Let $n, a_1, \cdots, a_n$ be positive integers and $p_1, \cdots, p_n$ be polygonal numbers; then the function $f=a_1p_1+\cdots+a_np_n$ is universal when $n=m+2$ and $a_1=\cdots=a_n=1$, and $p_3, \cdots, p_{m+2}$ may be taken to be 0 or 1 if $m>2$. In this paper the universal functions f are determined for $m\geqq 3$, $a_1+\cdots+a_n\leqq m+2$; the cases $m=1$ and 2 have been treated for any n by Liouville and Dickson.

3. Professor L. E. Dickson: *All universal ternary quadratic forms are null forms.*

A positive or indefinite form f is called universal if it represents all positive integers or all positive and negative integers respectively. For three variables every universal f is proved to be a null form, that is, f vanishes for integral values not all zero of the variables. Since all universal Null ternary forms were found in another paper presented by the writer at this meeting, this completes the theory of all universal ternary quadratic forms.

4. Professor L. E. Dickson: *All indefinite quadratic forms in 3 or more variables which represent every integer.*

Of ternary quadratic forms, no positive form represents every positive integer, but infinitely many indefinite forms are universal, that is, represent every positive, negative, and zero integer. All universal $ax^2+by^2+cz^2$ are determined explicitly. A method is developed to find all Null universal quadratic forms in n variables. The results are complete for $n=3$ and essentially complete for $n=4$. For $n\geqq 5$, the theory is sufficient but involves more computation.

5. Professor F. E. Wood: *Cubics whose (Hessian)^n are themselves.*

This paper appears in full in the present issue of this Bulletin.

6. Professor K. E. Rosinger: *A note concerning the symbols ϕx and $\phi\hat{x}$.*

It is the purpose of this paper to examine the basis of the distinction made in *Principia Mathematica* between the symbols ϕx and $\phi\hat{x}$. The problem is considered in the light of Frege's studies in the logical analysis of functions. It is found that the distinguishing characteristic of propositional functions recognized by Whitehead and Russell, namely "ambiguity," is derived from a more fundamental notion, which Frege calle Ergänzungsbedürftigkeit. This notion is represented by Principia's symbol ϕx. On the other hand, $\phi\hat{x}$ expresses Frege's concept Zusammengehörigkeit. It appears that neither Frege nor Principia have recognized that these two concepts are independent of each other. Hence the distinction between ϕx and $\phi\hat{x}$ is fundamental. The fact that this has not been detected by many mathematical logicians has produced considerable confusion. From this fundamental difference, the paper deduces a number of properties of the notions here considered, thereby bringing to explicit statement the meanings and purposes of ϕx and $\phi\hat{x}$.

7. Dr. H. S. Wall: *On extended Stieltjes series.*

This paper refers to the author's previous article which appears in the Transactions of this Society, January 1929, on Stieltjes series and continued fractions. The writer considers the problems of extending the Stieltjes series

$$\sum_{i=0}^{i=\infty} c_i(-z)^i \tag{1}$$

to the left by prefixing successively terms of the form $(-1)^k c_{-k} z^{-k}$, $k=1, 2, 3, \cdots, n$, in such a way that all of the series

$$\sum_{r=0}^{r=\infty} c_{-k+r}(-z)^r, \quad (k=1, 2, 3, \cdots, n),$$

shall be Stieltjes series. A necessary and sufficient condition for a first extension, given in the preceding article, is replaced by the following. Let the limit of the sequence of even convergents of the Stieltjes continued fraction corresponding to (1) be

$$F(z)=\int_0^\infty \frac{d\phi_1(u)}{1+zu} \tag{2}$$

Then (1) admits a first extension if and only if

$$\lim_{a\to 0}\int_a^\infty \frac{d\phi_1(u)}{u}, \quad (a>0)$$

exists and has a finite value k; and $c_{-1} \geqq k$. From this it follows that for a large class of Stieltjes series, including among others those for which the corresponding Stieltjes continued fractions are divergent, extensions of infinite order exist in which

$$c_{-k}=\int_0^\infty \frac{d\phi_1(u)}{u^k}.$$

The Stieltjes continued fraction corresponding to the Stieltjes series

$$\sum_{i=1}^{i=\infty}(-1)^{i+1}\frac{C_i}{z^i} \tag{3}$$

converges for all z not belonging to the interval $(0, -R)$, where R is the distance from the origin to the most remote pole of (2), to $F(z)$, and the series (3) converges for all $|z|>R$ to the function $F(z)$.

8. Professor R. L. Wilder: *A topological definition of simple closed surface.*

The following characterization of simple closed surface is obtained. In a metric separable space let M be a compact set containing at least one simple closed curve, and satisfying the following conditions: (1) If t is an arc of M, then $M-t$ is connected; (2) if J is a simple closed curve of M, then $M-J$ is the sum of two uniformly regular components. Then M is homeomorphic with the sphere $x^2+y^2+z^2=1$.

9. Professor R. L. Wilder: *Concerning the Phragmen-Brouwer Theorem.*

Let S be a topological space. Consider the following properties: I. If F_1 and F_2 are mutually exclusive closed subsets of S, neither of which separates S, then F_1+F_2 does not separate S. II. If N is a continuum in S and R a component of $S-N$, the boundary of R is a continuum. Either of these properties serves as a statement of the Phragmen-Brouwer Theorem if S is a euclidean space. It is the purpose of the present paper to study the equivalence between these properties, and certain related properties, as well as to determine certain spaces in which they hold true. Examples are given of spaces where I holds but II does not hold, and conversely. It is shown, however, that in a certain class of general spaces, of which euclidean spaces form a proper subset, the two properties are equivalent. The following generalization of the theorem of Kuratowski (Fundamenta Mathematicae, vol. 8, p. 149) is obtained: If, in a euclidean plane E_2, M is a bounded continuum containing more than one prime part and such that no one of its prime parts separates E_2, then in order that M may be considered as a space having property II it is necessary and sufficient that M should not separate E_2.

10. Professor V. G. Grove: *A canonical form of Green's projective analogue of the Gauss differential equations.*

In this paper the author develops a canonical form of the projective analogue of the Gauss differential equations. The vertices of the tetrahedron of reference are covariants of simple geometric significance. Each invariant appearing in the canonical form has a geometric significance easily read from the equations. By specializing the parametric net to be conjugate, one obtains the equivalent of Lane's canonical form for conjugate nets. By assuming the parametric net to be non-conjugate, a canonical

form for Green's differential equations of a non-conjugate net is obtained. In this case also the geometrical significance of the invariant coefficients may be seen directly from the differential equations.

11. Professor E. P. Lane: *On the fundamental transformation of surfaces.*

The author develops the theory of a fundamental transformation of surfaces in ordinary space, treating the subject projectively and using Wilczynski's method based on a completely integrable system of differential equations. The conjugate and harmonic invariants recently used by Slotnick receive new geometric interpretations. Certain quadric surfaces called the associated quadrics of the transformation are introduced into the theory. Various theorems are obtained concerning fundamental transformations with conjugate and harmonic congruences of special types.

12. Professor R. E. Langer: *On the asymptotic forms of the solutions of certain differential equations containing a parameter.*

This paper deals with the equation $y''+\lambda^2\phi(x)y=0$ in which $\phi(x)$ vanishes like x^k at $x=0$. It is shown that a certain approximate equation is solvable in terms of Bessel's functions, and that there exists in any upper or lower half λ-plane a pair of solutions of the given equation which for $0<\alpha\leqq x\leqq\beta$ have asymptotic forms with the same leading terms as the solutions of the approximate equation, and which are also represented asymptotically by the latter at $x=0$. By means of these solutions and a formula expressing the general solution $y(x)$ in terms of an arbitrary pair $y_1(x)y_2(x)$, one obtains the leading terms of the asymptotic expressions representing $y(x)$ for λ in different half-planes, or x in different intervals on opposite sides of the origin.

13. Professor L. E. Dickson: *New results and methods for minima of quadratic forms.*

In 1903 Markoff obtained the first three minima of all real indefinite ternary quadratic forms for integral values of the variables, and emphasized the difficulty of the problem of finding further minima. The fourth mimimum is now found by his method. But new methods are developed to obtain further minima. An application is to the tabulation of representatives of all classes of a given determinant. Such a table is used to obtain general theorems on the representation of all integers by such quadratic forms.

14. Professor M. H. Ingraham: *The reducibility of algebras without a finite base.*

Systems are considered which satisfy the usual postulates for a linear associative algebra except they may not contain a finite base. From the definition of a linear set as one which is closed under addition and scalar

multiplication, we find that the definition of the sum of a class of linear sets, as the minimal linear set containing every linear set of the class, has content. As Wedderburn has shown under slightly stronger hypotheses, most of the elementary theorems on the direct sum of linear sets follow with, at most, a slight change in the usual proof. Algebras exist containing no irreducible sub-algebras. If an algebra can be reduced into irreducible sub-algebras, each with a principal unit, the reduction apart from order is unique. An algebra with a principal unit cannot be expressed as the direct sum of more than a finite number of algebras. This last fact is partly due to restrictions in the definition of a sum rather than being inherent in the problem.

15. Professor E. T. Bell: *A partial isomorphism between the functions of Lucas and Weierstrass.*

This paper will appear in full in an early issue of this Bulletin.

16. Professor E. T. Bell: *A class of polynomials and rational functions in four variables.*

The polynomials and functions are those arising as coefficients o $1, i, j, k$ in the nth power, n an integer, of the quaternion whose coordinates are w, x, y, z. These functions have a number of interesting properties, of which the following will suffice as specimens. With respect to one and only one of the independent variables, the polynomials are Appell polynomials. With respect to a properly chosen argument, the polynomials are Tchebycheff polynomials. With respect to n, both the polynomials and the functions are Lucas functions. Both the polynomials and the functions, with respect to w, x, y, z, satisfy an extremely simple set of 4 self-adjoint differential equations of the second order. The polynomials are rationally resolvable into factors of degrees $\leqq 2$ in an elegant way. With respect to n they have addition and subtraction theorems abstractly identical with those for the circular, hyperbolic and Lucas functions. If the norm of the generating quaternion is unity, the polynomials and functions, in their algebraic aspects, are simply isomorphic with the sigma, zeta, and pe functions of Weierstrass.

17. Professor E. T. Bell: *A class of polynomials and rational functions in eight variables.*

These are the generalization of the preceding four-variable polynomials and functions to the like arising from the eight-square theorem of Degen and Cayley. The preceding abstract may be repeated verbatim with respect to these. No further generalization in the same direction is possible.

18. Professor W. A. Manning: *The degree and class of doubly transitive groups.*

In this paper it is proved that if μ is the class and n the degree of a doubly transitive group which contains a substitution of degree μ and of

prime order $p(>3)$, then $\mu>(n/2)(1-1/p)-(n^{1/2}/2)(1-1/p^2)^{1/2}-1$. The same limit holds when $p=3(\mu>3)$ if n is sufficiently large. This result complements the limit $\mu>n/2-n^{1/2}/2-1$, for $p=2$, published by the author in 1914.

19. Professor T. H. Hildebrandt: *Remarks on Carathéodory's theory of measure.*

Carathéodory bases his theory of measure on an upper measure function satisfying five postulates. This paper shows how from an upper measure function satisfying the first three postulates it is possible to construct an upper measure function satisfying the fifth, and obtains conditions under which the property of measurability is unaltered. The same study is undertaken beginning with an upper measure function satisfying only the first postulate.

M. H. INGRAHAM,
Associate Secretary

A CORRECTION

BY W. J. TRJITZINSKY

In my paper, *Zeros of a function and of its derivative,* (this Bulletin, Vol. 33 (1927), pp. 693-695), 2π should be replaced everywhere by $2\pi n$ (integer$\geqq 1$), except on line 7, page 693, and in expressions $1/(2\pi i)$.

THE INAUGURATION OF THE INSTITUTE HENRI POINCARÉ IN PARIS

BY MAURICE FRÉCHET

In November, 1928, there was formally inaugurated in Paris a new institute of mathematics. It was both the official opening of a new building and the beginning of new courses of lectures, all to be a part of the Faculty of Sciences of the University of Paris.

The building is now ready, but the internal arrangement and the furnishing will not be completed for some time. It was considered wise, however, to hold the ceremony in the building in order to attract public attention to the opening of the lectures and to the foundation of the Institute.

It was desired to express the gratitude of the University of Paris towards those who had provided the necessary means. The history of this Institute is brief. It had been noted by the International Education Board that several opportunities had led them to give very large sums of money to various universities in Europe, but that gifts to French universities had been on a much smaller scale. Noting the importance of the French Mathematical School, it was thought that helping mathematics in France was perhaps one of the best ways to help science all over the world.

The decision was taken after many consultations, in which Professor Trowbridge, then representing in Paris the International Education Board, and Professor Birkhoff as a great mathematician, took decisive part.

It was decided to ask Professor Emile Borel to draw up a plan. The plan, which was approved, creates under the name INSTITUT HENRI POINCARÉ a center for teaching and for research on mathematical physics and on the calculus of probabilities.

The new teaching positions have been given to three men.

The courses on *Physical Theories* will be delivered by Professor Léon Brillouin and M. Louis de Broglie (to be distinguished from physicists of the same names, both members of the Académie des Sciences). Professor Léon Brillouin has made himself known by his fundamental researches on the theory of quanta and its applications. He was called last year to expound them in several universities of the United States and Canada. Dr. Louis de Broglie is the creator of that wave mechanics which, though created only yesterday, now plays a leading part in mathematical physics, and has been the inspiration of many investigations of revolutionary nature.

Although these courses will form an important addition, those interested in theoretical physics will know that there were already important courses on this subject in Paris, among which should be mentioned those of Professor Brillouin and Professor Langevin at the Collège de France, and those of Professor Eugène Bloch and Professor Villat at the Sorbonne.

As for the calculus of probability, it had already its great exponent at the Sorbonne in Professor Emile Borel. His researches on this subject and his personal action have done much to revive in France the interest in this science which owes so much to such French scientists as Pascal, Fermat, Laplace, Poisson, Bienaymé, Cauchy, Cournot, Bertrand, Henri Poincaré.

To Professor Borel's course will now be added a new course by Maurice Fréchet, formerly professor at the University of Strasbourg. His theory of abstract spaces and functions has already made him known in America, where he was called to expound it at the University of Chicago in the summer quarter of 1924. But of late he has devoted much attention to the theory of probability, on which he published (in collaboration with Professor Halbwachs) a work entitled *Le Calcul des Probabilités à la Portée de Tous.*

It should be stated also that the applications of proba-

bilities to social sciences are taught in the already existing Institut de Statistique of the University of Paris.

The activities of the Institute Henri Poincaré will not be confined, however, to the new courses. It aims to be international in scope. The attendance at these courses is very cosmopolitan indeed. But the Institute will also have an international staff of lecturers. In addition to the permanent courses, single lectures or brief series of lectures will be given by distinguished scientists. Professors Vito Volterra of Rome, and de Donder of Bruxelles, have already promised their cooperation. Other engagements will soon be published.

Finally, as the ever-increasing numbers of lecturers and students at the Sorbonne called for new measures, it was decided to seize upon the opportunity and to erect a new building where not only the new courses, but all the advanced courses on mathematics will be given, and where the mathematical library will be housed. The International Education Board is to contribute one hundred thousand dollars to these expenses; Baron Edmond de Rothschild also contributed twenty-five thousand dollars, and the French Ministry for Education 30,000 francs.

It is to be hoped that some of the students and scholars who plan to complete their scientific education or to pursue their researches in Europe, will remember that there has been created in Paris, thanks chiefly to American generosity a great international scientific center for mathematical physics and the calculus of probability.

MAURICE FRÉCHET

REPORT ON THE 1928 INTERNATIONAL CONGRESS OF MATHEMATICIANS*

BY L. TONELLI

At the Toronto Congress of 1924, Professor Salvatore Pincherle was elected president of the International Mathematical Society and the city of Bologna was chosen as the seat of the 1928 congress. A program committee was selected, the presidency being entrusted to Senator G. Albini, Rector of the University of Bologna. An executive sub-committee was also formed with Professor S. Pincherle as president, the other members being: Professor E. Bompiani, Professor P. Burgatti, Professor F. Guarducci, Professor G. Horn d'Arturo, Professor Q. Majorana, Professor U. Puppini, Professor L. Tonelli, Professor D. Zucchini, Comm. G. Borsari (treasurer) and Professor Ett. Bortolotti (general secretary). The congress was held under the distinguished patronage of H. M. the King of Italy, and the honorary presidency of His Excellency, Benito Mussolini, the leader of the government.

The program committee, following the wishes of the majority of the Toronto delegates, invited to the congress all mathematicians, and representatives of the leading universities and academies of the world. The proceedings of the congress were divided into seven sections, and several chairmen were named for each section, their task being the detailed organization of their particular section. The arrangement of sections was as follows:

SECTION I. Arithmetic, Algebra, Analysis (Chairmen: M. Cipolla, G. Fubini, G. Scorza, L. Tonelli).

SECTION II. Geometry (Chairmen: L. Berzolari, E. Bompiani, G. Fubini, F. Severi).

SECTION III. Mechanics, Astronomy, Geodesy, Geophysics,

* Translated by Dr. A. O. Hickson, Brown University.

Mathematical Physics, Theoretical Physics (Chairmen: G. Armellini, P. Burgatti, E. Fermi, T. Levi-Civita, C. Somigliana).

SECTION IV. Statistics, Mathematical Theory of Economics, Calculus of Probability, Actuarial Science (Chairmen: L. Amoroso, P. Cantelli, C. Gini, G. Toja).

SECTION V. Engineering and Industrial Applications (Chairmen: G. Albenga, C. Di Pirro, G. Giorgi, F. Lori, M. Panetti, C. Porro, U. Puppini, L. Silla, G. Vacchelli, G. C. Vallauri).

SECTION VI. Elementary Mathematics, Didactics, Mathematical Logic (Chairmen: U. Amaldi, G. Peano, A. Perna).

SECTION VII. Philosophy, History of Mathematics (Chairmen: E. Bortolotti, F. Enriques, R. Marcolongo, G. Vacca).

Eight hundred active members (those mathematicians who took part in the proceedings of the congress), 287 associate members (those belonging to the families of the delegates), and most of the members of the program committee attended the congress. Thirty-four nations were represented, and the larger delegations, except that of Italy (249 members), were: Germany with 76, Great Britain and Dominions with 64, France with 51, the United States with 43, Switzerland with 33, Poland with 32, Russia and Ukraine with 35, Hungary with 22, etc.

On Sunday evening, September 2, the Italian Mathematical Society held a reception for the delegates in the hall of the Bologna Circolo di Cultura. On Monday morning, September 3, the inaugural meeting was held in the assembly hall of the Archizinnesio in the presence of H. R. H., the Duke of Bergamo; the speakers were: Mayor Arpinati of Bologna, Senator Albini, rector of the University, Professor Pincherle, Professor Birkhoff, representative of all the foreign mathe-

maticians, and S. E. Belluzzo, minister of Public Instruction. The last speaker formally opened the congress.

The proceedings of the congress began on the afternoon of the same day and all meetings were held in the halls of the Royal University (Via Zamboni). At the first session a general assembly was held for the nomination of the president and vice-presidents; the following nominations were made by acclamation: Professor S. Pincherle for president of the congress, and for vice-presidents Professors de la Vallée Poussin, J. Hadamard, D. Hilbert, W. H. Young, J. C. Fields, O. Veblen, E. Terradas, W. Sierpinski, H. Bohr, N. Lusin, S. Kakeya.

The general lectures, all of which were given in the mornings, were successively as follows:

D. HILBERT—*Probleme der mathematischen Logik.*

J. HADAMARD—*Le développement et le rôle scientifique du calcul functionnel.*

U. PUPPINI—*Le bonifiche in Italia.*

E. BOREL—*Le calcul des probabilités et les sciences exactes.*

O. VEBLEN—*Differential invariants and geometry.*

G. CASTELNUOVO—*La geometria algebrica e la scuola Italiana.*

W. H. YOUNG—*The mathematical method and its limitations.*

V. VOLTERRA—*La teoria dei funzionali applicata ai fenomeni ereditari.*

H. WEYL—*Darstellung kontinuirlichen Gruppen.*

T. VON KÁRMÁN—*Mathematische Probleme der modernen Aerodynamik.*

L. TONELLI—*Contributo italiano alla teoria delle funzioni di variabili reali.*

M. FRÉCHET—*L'analyse général et les espaces abstraits.*

R. MARCOLONGO—*Leonardo da Vinci nella storia della matematica e della meccanica.*

N. LUSIN—*Sur les voies de la théorie des ensembles.*

L. AMOROSO—*Le equazioni differenziali della dinamica economica.*

The number of sectional papers, all of which were pre-

sented in the afternoons, reached the remarkable total of almost four hundred.

A reception by the Mayor, a reception by the National Government, an orchestral concert of historical Italian music, and a banquet by the program committee of the congress occupied the evenings.

September 7 was devoted to a tour of the cities of Ravenna and Ferrara, and Lake Ledro.

Monday morning, September 10, the last day of the congress, the delegates departed for Florence, where, in the Palazzo Vecchio, Professor Birkhoff spoke on *Quelques éléments mathématiques de l'art.* The final session of the congress was also held here, and it was decided, completing the discussion of the preceding day, that the congress of 1932 should be held in Switzerland.

During the seven day visit of the delegates at Bologna and the closing day of the congress at Florence, everything proceeded in perfect order, and the best of relations existed between the delegates of different nationalities. To the scientific interest, determined by the interchange of ideas, of methods and of results on the most modern tendencies of mathematics, one must add the result, politically important, of bringing the international congress of mathematicians to the universality of the pre-war period.

L. TONELLI

MAXIME BÔCHER*

BY WILLIAM F. OSGOOD

The twelfth of last September marked a decade since Bôcher passed away. He retained the full vigor of his mind till shortly before the end, but the last months of his life were clouded by a wasting disease, to which he finally succumbed.

Maxime Bôcher was born in Boston, August 28, 1867. He prepared for college at the Cambridge Latin School and graduated from Harvard in the Class of '88. He then went to Göttingen, where he spent six semesters and took the degree of Doctor of Philosophy in 1891. He was appointed instructor in mathematics at Harvard and entered on his duties in the fall of that year.

To understand Bôcher as a teacher, we must first look at mathematics in Cambridge in the late eighties. The traditional regard in which mathematics had always been held in New England was enhanced by the influence of Benjamin Peirce, a rugged figure in the scientific life of his time. With Byerly and B. O. Peirce came not only more effective instruction, but a systematic organization of basal courses in the calculus, in geometry, in mechanics, and in mathematical physics. Cole had just brought from Leipzig Klein's geometric treatment of the theory of functions of a complex variable, and his enthusiasm for the mathematics that had been revealed to him in Klein's seminar—for algebra and geometry, even more than for analysis—predisposed Bôcher to seek the guidance of that great leader.

On his return to Harvard as instructor he was entrusted with an elementary course. I use the word *entrusted* advisedly; for although Professor Byerly was willing that any one who was master of his subject should teach graduate stu-

* An address delivered before the Society at the request of the program committee, New York, December 28, 1928.

dents, he well knew that a beginner must be a born teacher if he is to teach beginners effectively. And Bôcher was a born teacher. On his mother's side he was of Puritan ancestry, and it is hard to imagine a Puritan mother who had not the call to teach. His father was professor of French, whose courses, because of the joy Professor Bôcher felt in the beauty of literature, made a wide appeal to the undergraduates of that day. Of the sternest scientific conscience, the younger Bôcher was keenly aware that, though truth may not be relative, the ability to recognize it is. With his broad and deep knowledge of his subject, whether he was teaching analytic geometry to freshmen or linear differential equations to graduates, he sought first of all to understand the student's domain of ideas, and then to give him the best which he could receive. It is generally recognized that interest in the science of mathematics and productive scholarship in that field redound to the good of elementary instruction. Bôcher proved the converse. Because he entered into the intellectual life of the freshman and devised ways and means to guide him in the pursuance of mathematics, he was thereby enabled to make his graduate instruction effective to an astonishing degree. When the Annals of Mathematics came to Cambridge, with Bôcher as the first editor, he sought and secured collections of problems in advanced subjects, results of such teaching as I have indicated in the higher fields. He encouraged, too, the preparation of expository articles.

He understood as few teachers do, the value and the difficulty of formulation. Think what a change would spread over the face of mathematical literature if writers had any adequate conception of formulation and the role it plays in clear thinking, and if they had the tact required to show them the relation of their message to the reader's knowledge. For most mathematicians are not intentionally mystics, either in the class-room or in the written word. They fail in the first requirement which science imposes on its followers, namely, to appreciate the importance of stating the probem definitely

and of seeing clearly the relation of the problem or theory to the rest of mathematics. I say, to *appreciate* the importance; for no improvement can come in presentation, whether it be a question of the author's own discoveries or of a college text-book, till writers *feel* the importance of clear formulation of results and of methods, and of clear insight into the inter-relations of the parts. It was this emotional element in Bôcher that at once made him a master in the class-room and directed his own energies into fruitful fields of research.

His *Algebra* was an outgrowth both of his teaching and of his own scientific work. When he gave his first advanced course in algebra (in 1898) there was no systematic treatment, in any language, of those parts of algebra which are most useful in their applications to geometry, to mathematical physics, and particularly to the calculus and analysis. The need of the student in this direction was clear to Bôcher, for he had himself, as a student, been obliged only recently to provide himself, unaided, with these tools. A paper on *Linear dependence and linear equations*, in the Annals of Mathematics for 1901, was his first published work in this field. It opens with the words:

> The subject treated in the following pages is not only one which, owing to its numerous applications, is of considerable importance, but, while distinctly elementary, illustrates one of the most striking tendencies of modern algebraic and analytic work, namely, the tendency not to be satisfied with results which are merely true "in general," i.e., with more or less numerous exceptions, but to strive for theorems which are *always* true.

And his understanding of the needs of the reader is shown by the footnote which he appends to this section:

> The great importance of this tendency will be apparent if we remember that when we apply a theorem, it is usually to a special case. If we know merely that the theorem is true "in general," we must first consider whether the special case to which we wish to apply it, is not one of the exceptional cases where the theorem fails.

Although his personal needs had not, so far as I am aware, led him to treat the factoring of polynomials in several variables, he recognized the importance of this subject, and his chapter on this topic gives precisely the information

which is needed for the study of the analogous problem of factoring analytic functions of several complex variables in the neighborhood of a point. Out of the systematic study of this latter problem has grown a theory of elimination which makes possible the proof of Weierstrass's second theorem on implicit functions in a single page.

The finish, both in form and substance, with which this book is written, can easily present a danger to the beginner, who has not yet learned that he must write his own book if he is to master the methods. But Bôcher's great skill in devising exercises, small and great, serve as a corrective for ill-founded self-confidence. As a sample of the kind of demand Bôcher encouraged his students to make on themselves, I may mention his paper in the Annals of Mathematics (October, 1905): *A problem in analytic geometry with a moral.*

Bôcher's first book, *Ueber die Reihenentwickelungen der Potentialtheorie*, appeared in 1894, and was a further development of the subject of his doctoral dissertation and prize essay. A perusal of the table of contents shows the fields with which he had become familiar: not projective geometry alone, but the geometry on a quadric manifold in higher spaces, with the method of elementary divisors for purposes of classification; the cyclides of Dupin and curvilinear coordinates, with extension to space of n dimensions; the partial differential equations of mathematical physics, leading him in his later work to consider that which is vital in integral equations and eventually to write a Cambridge Tract on this subject; total differential equations of the second order, linear and homogeneous, considered both in the complex domain and in reals, with reference to the theorems of oscillation of Sturm and Klein; Lamé's equation. It was in his later lectures on this part of the subject that one of his most gifted pupils obtained his start in mathematics. The development of arbitrary functions into series of standard types; systems of orthogonal functions, preparing the way for his later guidance of a student in the field of divergent

series and convergence factors, out of which studies have grown valuable contributions to this important subject.

It may seem that Bôcher imitated Klein, or that he was an adherent of a Göttingen School. But ideals cannot be imitated, and these ideals, in particular, are the spirit of the thought of free men, striving for a higher and deeper sense of scientific values in broad fields of mathematics.

Bôcher gave, at one time or another, nearly every one of the back-bone courses in the Department; but his special interest lay in the comprehensive course on modern geometry, to the development of which he contributed much, and on which, at the time of his death, he had begun to write a book. He gave, furthermore, for a number of years, one or the other of the introductory courses in the theory of functions of a complex variable, and of real variables. After Cole, he was the first at Harvard to give the former course from the geometric standpoint of Klein.

The latter course originated with him, being entitled *Infinite Series and Products*.* In the literature of that time every rigorous proof was given by ϵ's, *ab ovo*. The fundamental theorems of analysis must, of course, be established by means of ϵ-proofs. But the derived theorems stand out more clearly when deduced from the former. It was not a question of logical elegance or of a doctrinal method, but rather of putting first things first, and helping the student to grasp the inherent difficulties of higher analysis. Bôcher had contempt for all that smacked of platform erudition, the cult peculiar to those who had a sense for rigor, of stating every theorem with the maximum of generality, at the expense of incisiveness with reference to its essential meaning.

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* On looking up the date I find that I gave this course for the first time in 1897–98. The error is interesting, for it shows the close touch in which Bôcher and I were in organizing these courses. Bôcher had, for a year or two, brought into a half-course on the theory of equations and invariants a small amount of algebraic analysis, and he suggested the desirability of a half-course on infinite series and products—a suggestion in which I heartily concurred.

His advanced course was frequently on total linear differential equations of the second order, alternating between the complex theory and the real case. He also lectured repeatedly on the partial differential equations of mathematical physics, and on integral equations. Out of these courses grew his address at the Cambridge Congress (1912), and his Paris lectures (1913–14).

Throughout his whole academic career, beginning while he was still an instructor, he was actively engaged in guiding students in their work for the doctorate, and about twenty of these students completed their work under him. He was extraordinarily successful in discerning important topics, with range and depth, yet still within the powers of students at that stage of advancement. He was skillful in his guidance, not giving help that hampered the student's initiative or resourcefulness, nor neglecting him when he was ready to profit from a conference.

As editor of the Transactions he took a similar interest in the work of the younger mathematicians. His instinct for square dealing was shown anew by his impartial consideration of each paper submitted and his insistence on a prompt acceptance or rejection, followed in the former case by prompt publication. No referee could delay action long without hearing from Bôcher, and further delay caused the paper promptly to be recalled. He was no respecter of persons. Each paper was judged on its merits, and an opportunistic attitude toward contributors would have seemed to him sacrilege. He took great pains in helping unskillful writers to improve their presentation, but his first thought was for the genuineness of the scientific results obtained. Thus he contributed in those early days to make the traditions of the Transactions high in relation both to form and to substance, and to give the journal a reputation for fairness alike toward the older and the younger writers.

His work on the Annals had given him a valuable apprenticeship. It would have been easy to make the journal, when it came into the hands of the younger

mathematicians, interested as we all were in making for the Transactions the highest possible place in the mathematical world, a repository for papers not quite good enough for the latter journal. This was not Bôcher's view of the province of the Annals. A paper that had little to recommend it but the novelty of the result and the avoidance of logical errors in the proof would not pass muster. He set the same high standards of scientific merit which characterized the more advanced journal. The scope of the paper might be restricted, but not the genuineness of the results.

Nor were his sympathies and interests confined to his own courses and his special fields of study. At a time when mathematicians were just beginning to turn their attention to postulates, he recognized in the work of Huntington investigations of substantial character and used his influence to secure for Harvard the latter's services. Again, it was with full consciousness of the hearty moral support Bôcher accorded that Bouton built up his two courses on the Lie theory, and developed them to an extraordinarily high degree of effectiveness. He was well informed concerning the character of Coolidge's work, though he had no great technical knowledge of the subject, and I recall the keen interest he felt in it when Coolidge was a beginner. The same is true of his interest in the work of each of the others of his colleagues in the department,—Birkhoff, Jackson, Graustein,* and Gabriel Green. It is due in no small measure to these qualities in Bôcher, to his sense of justice and honesty of thought, to his willingness to accord others, irrespective of age or rank, what it seemed to him reasonable that they should accord him under like conditions, that our department has been able to pass from a patriarchal system to a pure democracy.

The student of mathematics would, indeed, be temerous, who would ignore the past and, like some modern novelists, attempt to build up a civilization out of his inner conscious-

* Professor Graustein was an instructor in the department for the year 1913–14.

ness. True, the seer has visions of what man may accomplish. But the visions which light the way to new achievement are those which spring from an understanding of what man has accomplished. They are not the dreams of a child, just getting its first glimpses of the world.

Bôcher sank his roots deep into the past of his science. He was a student of history, for he found in it an aid toward perspective, toward seeing mathematical truths in their relative meaning and their relative importance, and toward putting first things first. He shunned ill-defined speculations and held rather to the method of research in the natural sciences—that of defining precisely an important problem and bringing the knowledge of the day to bear on its solution. He may sometimes have been too little attracted by the desire to make a brilliant guess; but what he built is a part of the permanent structure of mathematics.

Creating for the mere sake of creating was a thing in which Bôcher did not believe. Productive scholarship he did believe in, and his whole life gave sustained evidence of this cardinal point in his scientific faith, this corner stone in his scientific character. But productive scholarship meant to him something larger and grander than the discovery of new truth. For new truths come to us in unassorted lots, and because a theorem is true, it does not follow that it has value. Productive scholarship meant to him a deeper insight into mathematics. And the deeper insight comes not merely from the new theorem, but chiefly from its contacts. Bôcher was keenly aware that new truth is under suspicion; that the great majority of new theorems which can not stand the test of usefulness, are unimportant, spurious, and will not survive the nurturing care of their discoverers. He strove rather to advance mathematics according to a well-defined program, refusing to be attracted by novelty for novelty's sake, when the science was in the greatest need of coordination of great truths already obtained and of deepening the understanding of important results as yet but imperfectly perceived.

For this reason Bôcher's work was not spectacular. It was basic. For example, in reporting on his scientific contributions, Professor Birkhoff remarks that none of Bôcher's published papers lay properly in the theory of functions of a complex variable. And yet, after the founders of analysis have been cited, no name appears more frequently in the footnotes of the speaker's treatise than that of Bôcher.

Perhaps nowhere in the domain of human thought is a cultural background more important than in mathematics. To train one's mind to recognize the great structures man has built, the great visions human thought has seen, in the realm of the greatest of the sciences, it is helpful to have first-hand knowledge of achievements in other fields of human thought and human action. As an undergraduate Bôcher informed himself of the then recent discoveries of Darwin—not as the philosopher or theologian conceives them, but as the scientist knows them—experiments with well defined boundary conditions; conclusions exhausting the results of the experiments; speculations suggesting further study, but not accepted as truth till rigorously established.

All through his life Bôcher turned to literature for recreation. He was fond of Shakespeare, of the German and the French classics, and of biography. Though not a performer, he had a good appreciation of music and derived much enjoyment from listening to the production of classical music. And what was true of his intellectual life outside the science was eminently true within it. His sympathies were broad and the amount of accurate information he possessed in various fields of mathematics extensive.

So it came about that Bôcher was able to render a great service to mathematics in America through his sense of values. Now, a sense of values is not something that is inherited, like the artistic temperament. It is an expression of character. Character is formed in the determined and sustained struggle for ideals. But if the ideals are not intelligently chosen; if one has not taken the pains to form

consciously a certain philosophy of life; to detach oneself from the present and inquire whence and whither; to estimate that which is great and to plan how one may strive intelligently for the highest which, under the given limitations of temperament and ability, one may hope to attain, the result must be mediocre. I never knew a time in Bôcher's life in which he was not working on the most important problem of those among which he could reasonably expect to make advance.

It was this habit of mind, these standards relating to scientific activity, accompanied as they were by his striving for the highest and the purest beauty in mathematics, that made him helpful to others. He wished to see the influence of the Transactions, particularly on the younger writers, directed into these channels, and in this I am well aware that he had the earnest cooperation of our colleague and President, Van Vleck. What he lacked was the zeal of the reformer, who believes that men can be made to improve their ideals by having the truth thrust at them. He gave freely of himself to all who, in the spirit of truth, sought to discuss the truth with him. He felt no inner urge to make men do what of their own initiative they could not. Among the strongest of his contemporaries, however, his influence was strong, and he found hearty support from some of the best men of his time. His pupils have rendered a good account of themselves, and he is remembered by them as an inspiring teacher, not through emotional appeal, but through the genuineness of his mathematical perceptions and ideals, his sincerity, the rugged honesty of his thought, and his unswerving devotion to his science.

I may seem to have laid too much stress on Bôcher's appreciation of the work of others and his efforts to clarify and unify it, and the impression can easily be created that he absorbed knowledge rather than that he smote from the rock the living waters, that he had imagination of high order, and that he produced original results of high merit. It is not easy to correct this impression by citing specific results readily in-

telligible to the lay mathematician. All of Bôcher's more important work was based on theories which had already attained a high degree of advancement. To understand his precise contribution, it is often necessary to make a comprehensive study of a whole field, or, indeed, of different fields, of mathematics. But then it will be found that he brought an originality of thought and method which secure for him a high place among the mathematicians of his time. His scientific biographer, Professor Birkhoff, closes his estimate with the words:

"In amount and quality, his production exceeds that of any American mathematician of earlier date in the field of pure mathematics."

The contemplation of Bôcher's life leads us to the question: Is mathematics a collection of concepts and relations between these concepts—a *logical system*, in the broadest interpretation of this notion, as formulated by Bôcher himself in his St. Louis address—or is not a truer picture that of the various civilizations through which the human race has passed and is continuing to pass? These civilizations develop gradually, often at great expense, a purer conception of beauty, a deeper insight into truth, a perception of the eternal verities, a recognition of the absolute invariants of the human mind and of the human soul. And when these invariants have once been discerned, they pass over into the traditions of that people. They become a part of the intellectual and the spiritual heritage of the race.

Bôcher valued a new result, but he did not find the satisfaction in a collection of new results, or even new theories, which the child finds in a stamp collection. He looked beyond the new result to its place in the science, and his powers of criticism were such that he saw with remarkable clearness the bearing of that which is new on the development of the science.

Critical is a treacherous word to apply to a scholar, for it suggests destructive criticism and the cavilling of literary men. No greater mistake could be made than to think of

Bôcher as belonging to that category of critics. Constructive in all his thinking, he made his criticism of the work of others productive of new achievement on their part. He knew, too, how to profit from criticism of his work by others, even though given in an unfriendly spirit. It was the content, not the form, to which he chose to direct his attention.

We have witnessed the revolt of youth in society, the tearing of the mask from the conventions of the past, in these days since the War, and some of the older ones of us have felt humbled at the thought of our failure to make clear to those who come after us the eternal verities which are the heritage of the race. But each new generation must find these truths, and if it will not learn from the past, then it must create again the experience on which the wisdom of the past was founded. For build man must, whether for the hour or for eternity.

No less in mathematics have we a rich heritage from the past, and the desire of youth to go forward. The great problem is to stir the soul of youth to understand the former, while retaining in full measure the latter. That which Bôcher has left us has permanent value in this great field of human endeavor.

Bôcher's influence on American mathematics will wax with time, if we believe in progress. His influence is needed today. Not in economics and in ethics alone are the results of the Great War manifest. There is in mathematics an absence of scientific tranquility, a desire to get results without paying the price of scientific productivity. It is with satisfaction and inspiration that one turns to Bôcher's writings and Bôcher's teachings. The younger generation of mathematicians is not lacking in those who understand the message, and to them we may confidently entrust the future of mathematics in America.

We look to Bôcher as an example of one whose highest satisfaction was in service, but whose conception of service was derived from the highest needs of the science and of its disciples. To understand before undertaking; to examine

objectively the products of imagination; to make that which is genuine accessible for those who come after—these are the three great traits of his intellectual being.

A tribute to Bôcher would be imperfect which did not lay emphasis, not only on his high idealism and his intellectual achievements, but also on the character of the man, his passion for truth, his freedom from self-seeking, his high conception of duty and his loyalty to his fellowmen, above all, his faithfulness in little things. To him the words apply:

> How happy is he born or taught
> Who serveth not another's will;
> Whose armor is his honest thought,
> And simple truth his highest skill;
>
> Whose passions not his masters are;
> Whose soul is still prepared for death,
> Untied unto the world by care
> Of prince's ear or vulgar breath;
>
> Who hath his life from rumors freed,
> Whose conscience is his strong retreat,
> Whose state can neither flatterers feed,
> Nor ruin make oppressors great;
>
> Who God doth late and early pray
> More of his grace than goods to lend;
> And walks with man, from day to day,
> As with a brother and a friend.
>
> This man is freed from servile bands
> Of hope to rise, or fear to fall;
> Lord of himself, though not of lands,
> And having nothing, yet hath all.

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ON REGULAR POINTS OF CONTINUA AND REGULAR CURVES OF AT MOST ORDER n*

BY G. T. WHYBURN

1. *Introduction.* In this paper it will be shown, as a consequence of some more general results, that if n is any integer $\neq 2$, the set of all points of order n of any continuum M in a locally compact metric and separable space is punctiform, that is, contains no continuum; and hence there exists no continuum every point of which is of order exactly n.

The ordinary notation and terminology of point set theory will be employed. For example, $\overline{X}=X+X'$, where X' is the set of all limit points of the set X; $K\cdot H$ means the set of points common to K and H; $K\subset H$ means that K is a subset of H and $K\supset H$ that K contains H; $\delta(M)$ denotes the diameter of the set M; $\rho(X, Y)$ denotes the minimum distance between the sets X and Y; if R is an open set, $F(R)$ denotes the boundary of R relative to the whole space, and if R is an open subset of a set M, $F_m(R)$ denotes the boundary of R relative to M, that is, the set of all those points of $M-R$ which are limit points of R. By a continuous curve is meant any connected im kleinen continuum. A neighborhood of a point is an open set containing that point. A point P of a continuum M is called a Menger regular point† of M, or simply a regular point of M, if for each $\epsilon>0$, P can be ϵ-separated‡ in M by some finite subset of M, that is, a finite subset U of M exists such that $M-U=M_1+M_2$, where M_1 and M_2 are mutually separated, $M_1\supset P$, and $\delta(M_1)<\epsilon$. If an integer n exists such that, for each $\epsilon>0$, the ϵ-separating set U can be chosen of power n, but cannot, (for every ϵ), be

* Presented to the Society, Southwestern Section, December 1, 1928.

† See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306.

‡ P. Urysohn, *Sur la ramification des lignes Cantoriennes*, Comptes Rendus, vol. 175 (1922), p. 481.

chosen of power $<n$, P is said to be a point of order n of M. A continuum all of whose points are regular points is called a Menger regular curve, or simply a regular curve; and a continuum M such that for some integer n, all the points of M are of order $\leqq n$, is called a regular curve of at most order n.*

DEFINITION. The point P of a continuum M will be said to *separate locally* a given subset N of M in M if there exists a compact neighborhood G of P such that if R is any neighborhood of P lying in G, then $M \cdot \overline{R} - P$ is separated between some two points of $N \cdot R$, that is, $M \cdot \overline{R} - P = M_1 + M_2$, where M_1 and M_2 are mutually separated and $M_1 \cdot N \cdot R \neq 0 \neq M_2 \cdot N \cdot R$.

The point P of a continuum M is called a *local separating point*† of M if a compact neighborhood R of P exists such that $M \cdot \overline{R} - P$ is separated between some two points of the component C of $M \cdot \overline{R}$ which contains P. Obviously every local separating point of a continuum M *locally separates* M in M. However, the converse is not necessarily true unless P is a regular point of M.

2. THEOREM 1. *If for the closed subset N of a continuum M there exists an integer n such that the set K of points of N which are points of order $\leqq n$ of M and which locally separate N in M is dense in N, then the points of M of order $\leqq n/2+1$ are dense in N.*

PROOF. Let R be any open subset of N. By hypothesis $K \cdot R$ contains a point P_1. There exists a compact open subset R_1 of M containing P_1 and such that $N \cdot \overline{R}_1 \subset R$, $\delta(R_1) < 1$, $F_m(R_1)$ is of power $\leqq n$, and $\overline{R}_1 - P_1 = R_{11} + R_{12}$, where R_{11} and R_{12} are mutually separated and $R_{11} \cdot N \cdot R_1 \neq 0 \neq R_{12} \cdot N \cdot R_1$. There exists at least one of the sets R_{11} and R_{12}, call one such set D_1, such that $D_1 \cdot F_m(R_1)$ is of power $\leqq n/2$. Then since $F_m(D_1) \subset P + D_1 \cdot F_m(R_1)$, it follows that $F_m(D_1)$

* See K. Menger, loc. cit., and K. Menger, *Zur allgemeinen Kurventheorie*, Fundamenta Mathematica, vol. 10 (1927), pp. 96–115.

† See my paper, *Local separating points of continua*, presented to the Society October 27, 1928, appearing in Monatshefte für Mathematik und Physik, vol. 36 (1929), No. 2.

is of power $\leqq n/2+1$. Clearly $\overline{D}_1 \cdot N \subset R$ and $\delta(D_1)<1$. Let N_1 denote the set of points $\overline{D}_1 \cdot N$. By hypothesis, there exists a point P_2 in $K \cdot N_1 \cdot [\overline{D}_1 - F_m(D_1)]$. There exists an open subset R_2 of M containing P_2 and such that $\overline{R}_2 \subset \overline{D}_1 - F_m(D_1)$, $\delta(R_2)<1/2$, $F_m(R_2)$ is of power $\leqq n$, and $\overline{R}_2 - P = R_{21}+R_{22}$, two mutually separated sets with $R_{21} \cdot N \cdot R_2 \neq 0 \neq R_{22} \cdot N \cdot R_2$. For at least one of the sets R_{21} and R_{22}, call one such set D_2, it is true that $D_2 \cdot F_m(R_2)$ is of power $\leqq n/2$. Then, just as in the case of D_1, it follows that $F_m(D_2)$ is of power $\leqq n/2+1$ and that $\overline{D}_2 \subset \overline{D}_1 - F_m(D_1)$ and $\delta(D_2)<1/2$. Let $N_2 = \overline{D}_2 \cdot N$. Then $N_2 \subset N_1$, and there exists a point P_3 in $K \cdot N_2 \cdot [\overline{D}_2 - F_m(D_2)]$. Just as above it follows that there exists a subset D_3 of D_2 containing P_3 and such that $D_3 \cdot N_2 \neq 0$, $\delta(D_3)<1/3$, $\overline{D}_3 \subset \overline{D}_2 - F_m(D_2)$, and $F_m(D_3)$ is of power $\leqq n/2+1$. Let this process be continued indefinitely, giving a sequence $D_1, D_2, D_3, \cdots$, of subsets of M such that, for each i, $\overline{D}_i \subset \overline{D}_{i-1} - F_m(D_{i-1})$, $\delta(D_i)<1/i$, $0 \neq D_i \cdot N \subset R$, and $F_m(D_i)$ is of power $\leqq n/2+1$. There exists a point P which belongs to every set D_i. Then clearly P is a point of order $\leqq n/2+1$ of M. And since P is a limit point of N and N is closed, then P belongs to N and hence to R. Thus every open subset of N contains a point of order $\leqq n/2+1$ of M, and our theorem is proved.

COROLLARY 1a. *If for a continuum M there exists an integer n such that the local separating points of M of order $\leqq n$ are dense in M, then the points of order $\leqq n/2+1$ are dense in M.*

THEOREM 2. *If M is a Menger regular curve, and if for some integer n, all the local separating points of M are of order $\leqq n$, then points of M of order $\leqq n/2+1$ are dense in M.*

By a theorem of the author,* the local separating points in a regular curve M are dense in M. Then since every local separating point of M *locally separates* M in M, it follows by hypothesis and Theorem 1 that the points of M of order $\leqq n/2+1$ are dense in M.

* See my paper, *Concerning points of continuous curves defined by certain im kleinen properties*, offered to Mathematische Annalen.

COROLLARY 2a. *If for some integer n, every point of a continuum M is of order $\leqq n$, that is, if M is a regular curve of at most order n, then the points of M of order $\leqq n/2+1$ are dense in M.*

COROLLARY 2b. *There does not exist an integer $n \neq 2$ and a continuum M every point of which is of order exactly n.*

COROLLARY 2c. *The simple closed curve is the only continuum of at most finite order every point of which is of the same order.*

THEOREM 3. *If N is any connected set of regular points of a continuum M, the set K of all those points of N which are local separating points of M and locally separate N in M is dense in N.*

PROOF. Let R be any open subset of N, which we will suppose is $\neq N$, and let P be a point of R. Since P is a regular point of M, there exists a compact open subset D of M containing P and such that $F_m(D)$ is finite and $\overline{D} \cdot N \subset R$. Since N is connected and $D \cdot N \neq 0 \neq N \cdot (M-\overline{D})$, it follows that $F_m(D)$ contains a point X of N which is a limit both of $N \cdot D$ and of $N \cdot (M-\overline{D})$. And since N is connected im kleinen, it follows that there exist compact connected subsets A and B of N such that $A \cdot B = X$, $A - X \subset D$, and $B - X \subset M - \overline{D}$. Let G be a compact neighborhood of X such that $\overline{G} \cdot [F_m(D) - X] = 0$. Then if V is any neighborhood of X lying in G, $M \cdot \overline{V} - X = \overline{V} \cdot D + \overline{V} \cdot (M - \overline{D})$; and since D and $M - \overline{D}$ are mutually separated, it follows that $\overline{V} \cdot D$ and $\overline{V} \cdot (M - \overline{D})$ are mutually separated. Clearly these two sets contain points of A and B respectively. Therefore X is a local separating point of M which locally separates N in M, and hence it belongs to K. Thus, since $X \subset R$, it follows that K is dense in N.

THEOREM 4. *If M is any continuum and n is any integer >2, the set K^n of all points of M of order n is punctiform, that is, contains no continuum.*

PROOF. Suppose, on the contrary, that for some integer $n > 2$, K^n contains a continuum N. Now by Theorem 3, the set K of points of N which locally separate N in M is dense in N. Then, since every point of K is a point of order n of M, it

follows by Theorem 1 that N contains at least one point of order $\leqq n/2+1$ of M. But since $n>2$, the greatest integer which is less than or equal to $n/2+1$ is less than n. Hence N contains a point of order $<n$ of M, contrary to supposition.

NOTE. Menger (loc. cit.) has shown that the set K^1 of all points of order 1 (end points) of any continuum M is null dimensional, that is, each of its points can, for each $\epsilon>0$, be ϵ-separated in K^1 by a null set. Hence, using Theorem 4, if $n\neq 2$, the set K^n of points of order n of any continuum is punctiform. It would be interesting to determine whether or not, for $n\neq 2$, the set K^n is always null dimensional as well as punctiform. I will remark that the set K^w of points of order w (that is, regular points which are of no finite order) of a continuum, or even of a regular curve, is not necessarily punctiform, as can be shown by the curve obtained by making some simple additions to the Sierpinski regular curve.*

THEOREM 5. *In any regular curve M of at most order n, the set H of points of order $>n/2+1$ is null dimensional.*

PROOF. The set H must be punctiform. For suppose it contains a continuum N. Then since every point of N is a regular point of M of order $\leqq n$, by Theorem 3 the set K of points of N which locally separate N in M is dense in N. Hence, by Theorem 1, N contains a point of order $\leqq n/2+1$, contrary to supposition. Thus H is punctiform. And since by a theorem of Menger,† H is an F_σ (that is, the sum of a countable number of closed sets), it follows by a theorem of Mazurkiewicz‡ that H is homeomorphic with some linear set. Therefore, since H is punctiform, it is§ null dimensional.

We note here the fact that while the set H in Theorem 5 is null dimensional, it is not necessarily countable; for an

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* See W. Sierpinski, Comptes Rendus, vol. 160 (1915), p. 302.

† *Grundzüge einer Theorie der Kurven*, loc. cit., Theorem 4.

‡ S. Mazurkiewicz, Bulletin de l'Académie des Sciences de Cracovie, 1913.

§ See W. Sierpinski, *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 89.

example is easily constructed of a regular curve of at most order 3 in which the points of order 3 form a perfect set.

3. *Regular Curves of at Most Order 3.* Knaster and Kuratowski have shown* that the Sierpinski regular curve, which is a curve of at most order 4, contains a connected set which contains no perfect subset. In this section I shall show that no curve of at most order 3 can contain any such set, and indeed, that every connected subset of a continuous curve M, every maximal cyclic curve of which is a regular curve of at most order 3, contains a perfect set of local separating points of M. The continuous curve M is cyclicly connected† if every two of its points lie together on some simple closed curve in M. The cyclicly connected continuous curve C is said to be a maximal cyclic curve of a continuous curve M if C is a subset of M and is not a proper subset of any cyclicly connected continuous curve which is a subset of M.

LEMMA 6a. *In a regular curve of at most order 3, no two maximal cyclic curves have a common point.*

Lemma 6a follows at once from the facts that no two maximal cyclic curves of a continuous curve can have more than one point in common and that every point of a cyclicly connected curve C is of order not greater than 2 of C.

THEOREM 6. *If every maximal cyclic curve of a continuous curve M is a regular curve of at most order 3, then every connected subset H of M contains a perfect set of local separating points of M.*

PROOF. If every point of H is either a cut point or an end point of M, then‡ H is arcwise connected and hence contains

* *A connected and connected im kleinen point set which contains no perfect subset*, this Bulletin, vol. 33 (1927), pp. 106–109.

† See my paper, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38;for the extension of the results in this paper to a more general space than the plane, see an abstract by W. L. Ayres in this Bulletin, vol. 33 (1927), p. 661.

‡ See my paper, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194, Theorem 8; see also an abstract by C. M. Cleveland in this Bulletin, vol. 32 (1926), p. 420.

an arc every point of which is a cut point of M. If not, then H contains at least two points of some maximal cyclic curve C of M. Let $W=H\cdot C$. Then* W is connected and contains more than one point. Since, by hypothesis, every point of W is a point of order $\leqq 3$ of C, it follows by Theorem 3 that W contains a point P which locally separates W in C. Then since P obviously is a point of order $\geqq 2$ of W, it follows that a connected open subset R of C exists containing P and such that (1) $F_\epsilon(R)$ is of power $\leqq 3$, (2) $\overline{R}\cdot W$ is connected, and (3) $F_\epsilon(R)$ contains at least two points A and B of W which are separated in $\overline{R}$ by the point P. Now since $\overline{R}\subset C$, $\overline{R}$ is a regular curve of at most order 3. Hence, by Lemma 6a, no two maximal cyclic curves of $\overline{R}$ have a point in common. And since A and B do not lie in the same maximal cyclic curve of $\overline{R}$, it follows by a theorem of the author† that A and B are separated in $\overline{R}$ by each point of a perfect point set K. Now $K\subset W\cdot\overline{R}$, for $W\cdot\overline{R}$ is connected and contains both A and B. Clearly every point of K is a local separating point of M. Thus H contains a perfect set K of local separating points of M, and our theorem is proved.

It would be interesting to determine whether or not, in a regular curve of at most order 3, every connected subset is arcwise connected, and whether every punctiform subset is null dimensional.

The foregoing results yield immediately the following additional theorem on regular curves of at most order 3.

THEOREM 7. *In a regular curve M of at most order 3 the boundary with respect to M of every connected open subset of M is totally disconnected.*

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* G. T. Whyburn, loc. cit., Theorem 30.

† See my paper, *Concerning the cyclic elements of continuous curves*, presented to the Society September 4, 1928.

CUBICS WHOSE (HESSIAN)^n'S ARE THEMSELVES*

BY F. E. WOOD

The four equianharmonic, four degenerate and six harmonic cubics which belong to the syzygetic pencil $\lambda C+\mu H=0$, where $C=0$ is a general cubic and $H=0$ is its Hessian, have been studied in much detail. We denote them as the 14 *special* cubics of the pencil. It is known that the Hessian of a degenerate or equianharmonic cubic is a degenerate cubic, and that the (Hessian)^2, that is, the Hessian of the Hessian—of a harmonic cubic is itself and that the harmonic cubics are characterized by this property. The more general problem has been considered by Hostinský only, who entirely by calculation has obtained certain results for $n=3$ and $n=4$.† In this note a characterizing property for $n=3$ is obtained and certain general results noted, the method being quite different. The theorems of this note are believed to be new, though a part of Theorem 2 could be obtained from the work of Hostinský.

Denote by γ_n a cubic whose (Hessian)^n is itself and whose (Hessian)^t for every value $t<n$ is not itself. Let m denote the parameter of a cubic, and $m^{(n)}$ the parameter of its (Hessian)^n; then

$$m^{(r+1)} = -\frac{1+2(m^{(r)})^3}{6(m^{(r)})^2}, \qquad (r=0,1,\cdots,n-1),$$

where $m^{(0)}=m$. Then $m^{(n)}=m$ gives an equation in m of degree 3^n, which is also satisfied by any cubic whose (Hessian)^r is itself, for r a divisor of n. Dividing out from the equation the factors corresponding to these cubics, one will have in general an equation whose roots give cubics γ.

* Presented to the Society, December 31, 1928.

† B. Hostinský, *Sur les Hessiennes successives d'une courbe du troisième degré*, Proceedings of the International Congress, Cambridge, 1912, vol. 2, pp. 102–104.

None of the cubics involved can be special, when $n>2$. It can be shown that to every non-special cubic of a syzygetic pencil there are eleven others forming a group of twelve, each projectively equivalent to the others. The Hessians of such a group form another group of twelve projectively equivalent cubics, all distinct unless the group of Hessians is a group of special cubics; in which case for no value of n will the (Hessian)^n of one of the original group be itself.

If the group of Hessians is each time different from all the preceding groups, then the existence of one non-special cubic γ_n implies the existence of $12n$ such cubics. For every value of n the degenerate cubics will appear, and when n is a prime and greater than 2 the parameters corresponding to the degenerate cubics will be the only ones to appear in the equation, mentioned above, of degree 3^n. And one notes that 3^n-3, n a prime, is divisible by $12n$ when $n>3$. Thus *the totality of cubics γ_n form k groups of* 12 *projectively equivalent cubics each. When n is a prime and greater than* 3, *k is a multiple of n.*

Consider $n=3$. Let a_1 be a non-special cubic whose (Hessian)^3 is itself; let $a_2, \cdots, a_{12}$ denote the other cubics of the syzygetic pencil projectively equivalent to a_1, forming all together the group A; denote the Hessian of a_i by b_i, the group of b_i by B. Then there are two possible cases; either the group B is not, or is, the group A. In the first case one finds that there must be 36 cubics γ_3, which is impossible since for $n=3$, $3^n-3=24$. In the second case one finds that the continued Hessians of A form the group A, the twelve cubics of A being arranged in four groups of three, such that the continued Hessians of the cubics of any one triple are the triple itself.

Conversely, consider the non-special cubics which have the property that the Hessian of one such is projectively equivalent to it. Let m be the parameter of the cubic, then $-(1+2m^3)/(6m^2)$ is the parameter of its Hessian; hence we have the equation

$$\frac{m^3(1-m^3)^3}{(1-20m^3-8m^6)^2} = \frac{-\left(\frac{1+2m^3}{6m^2}\right)^3\left[1+\left(\frac{1+2m^3}{6m^2}\right)^3\right]^3}{\left[1+20\left(\frac{1+2m^3}{6m^2}\right)^3-8\left(\frac{1+2m^3}{6m^2}\right)^6\right]^2}.$$

Dividing out the factors $(1-20m^3-8m^6)^2(8m^3+1)^3$, to which correspond special cubics, we obtain an equation of degree 24. The 24 cubics corresponding are exactly the 24 previously mentioned and we have proved that *a necessary and sufficient condition that a non-special cubic have its (Hessian)³ coincide with itself, is that its Hessian be projectively equivalent to itself.*

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A PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA*

BY I. M. SHEFFER†

1. *Introduction.* The number of proofs given of the fundamental theorem of algebra is large. Perhaps for that very reason still another proof may not be unacceptable. The one that is offered here is not "elementary," since it makes use of some general results in analysis. Yet it may be termed simple, and may be of interest. It is our hope that this proof, which is believed to be new, may, with no great embarrassment, take its place in the family of proofs that every algebraic equation has a root.

2. *The Proof.* We consider the equation

(1) $a_kx^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0 = 0, \quad (a_k \neq 0, k > 0).$

There is no loss in generality in supposing that‡ $a_1 \neq 0$; and,

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† National Research Fellow.

‡ For suppose $a_1 = 0$. Make the substitution $x = y + \alpha$. We obtain an equation in y, in which the coefficient of y is a polynomial in α of degree

by dividing through by a_1, we can always make the coefficient of x unity. We then take $a_1 = 1$. Let us replace the constant term a_0 by $-\lambda$:

$$a_k x^k + a_{k-1} x^{k-1} + \cdots + x - \lambda = 0. \tag{2}$$

We shall prove that (2) has a root for every λ. For $\lambda = 0$ we have a root $x = 0$. Let us assume that x, considered as a function of λ, can be expanded in a power series about the origin (with no constant term):

$$x(\lambda) = \sum_{n=1}^{\infty} x_n \lambda^n. \tag{3}$$

If we substitute (3) into (2), and equate coefficients of like powers of λ, we find for $x_1, x_2, \cdots$, the series of equations

$$x_1 = 1, \quad x_2 = P_2(x_1), \quad x_3 = P_3(x_1, x_2), \cdots,$$
$$x_n = P_n(x_1, \cdots, x_{n-1}), \cdots,$$

where the P's are polynomials in their respective variables; and these permit us to determine, step by step, *and uniquely*, $x_1, x_2, \cdots, x_n, \cdots$.

The series (3) has a non-zero radius of convergence. One method of proof is to determine suitable inequalities for $|x_n|$. But it is simpler to appeal to the theory of implicit functions.

Denote the left hand member of (2) by $F(\lambda; x)$. We have $F(0; 0) = 0$, $\partial F(0; 0)/\partial x = 1 \neq 0$; consequently, in a sufficiently small neighborhood of $\lambda = 0$ (in the λ-plane), there exists a unique analytic function $x(\lambda)$ with $x(0) = 0$, which when put into (2) makes (2) an identity in λ. This function will have a convergent power series expansion, which by the uniqueness of the coefficients x_n, must coincide with (3).

For every λ inside the circle of convergence of (3), $x(\lambda)$ is a root of (2). If (3) converges for all* (finite)λ, then (2)

$k-1$, and since we know (by elementary algebra) that an equation of degree $k-1$ cannot have more than $k-1$ roots, it follows that α can be chosen so that this coefficient is not zero.

* This will be the case when $k = 1$.

has a root for every λ, and the desired theorem is established. Let us then consider the contrary case, where the radius of convergence, r, is finite; and let C denote the circle of convergence. On C, $x(\lambda)$ has at least one singular point. Let $\lambda=\lambda'$ be such a singularity.

Interior to C, $x(\lambda)$ *is bounded.* For suppose the contrary. Then there will exist an infinite sequence of values λ_n, with $|\lambda_n|<r$, such that $\lim |x(\lambda_n)|=\infty$. But $|\lambda|$ being bounded, for $|x|$ sufficiently large the dominant term in the left hand member of (2) is $|a_k x^k|$, so that for n sufficiently large, $x(\lambda_n)$ cannot satisfy (2); a contradiction.

It follows that there exists an infinite sequence λ_n, with $|\lambda_n|<r$, such that $\lim \lambda_n=\lambda'$ and such that $\lim x(\lambda_n)$ *exists.** Call this limiting value x'. Since $\lambda=\lambda_n$, $x=x(\lambda_n)$ satisfy (2), therefore x' is a root of (2) for $\lambda=\lambda'$.

We must then have $\partial F(\lambda; x)/\partial x=0$ for $(\lambda; x)=(\lambda'; x')$. For if not, then the implicit function argument tells us that in a sufficiently small neighborhood of $\lambda=\lambda'$ there is a unique analytic function $\bar{x}(\lambda)$ with $\bar{x}(\lambda')=x'$, such that $x=\bar{x}(\lambda)$ satisfies (2) identically. This function must be an analytic continuation of $x(\lambda)$.† But this means that λ' is not a singular point of $x(\lambda)$; a contradiction.

Now the condition $\partial F(\lambda; x)/\partial x=0$ does not involve λ; it is in fact an algebraic equation in x of degree $k-1$. Such

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* We use Weierstrass' Theorem on an infinite set which is bounded.

† A detailed argument is as follows: By the implicit function theorem there exist two positive numbers δ, ϵ such that for every λ in the λ'-region $|\lambda'-\lambda|<\delta$ there is one and only one x in the x'-region $|x'-x|<\epsilon$ which satisfies (2) (for $\lambda=\lambda'$ the value of x being x'); and this correspondence between λ and x in the regions considered constitutes an analytic function $\bar{x}(\lambda)$. Now the λ'-region overlaps with the region of convergence of (3). Hence for n sufficiently large we shall have λ_n in the λ'-region and $x(\lambda_n)$ in the x'-region, and this requires that $x(\lambda_n)=\bar{x}(\lambda_n)$. Now about this point $\lambda=\lambda_n$ (and for the value $x=x(\lambda_n)$), we can reapply the implicit function theorem. In a sufficiently small neighborhood of $\lambda=\lambda_n$ we find a unique analytic function of λ taking on the value $x(\lambda_n)$ at λ_n; and this function of course coincides with $\bar{x}(\lambda)$. But $x(\lambda)$ is analytic in a sufficiently small neighborhood of λ_n, and has there the value $x(\lambda_n)$, so that we must have $x(\lambda)\equiv\bar{x}(\lambda)$ in the neighborhood of λ_n.

an equation cannot be satisfied by more than $k-1$ values of x, so that (as we see from (2)) there cannot be more than $k-1$ singular points λ on C. We may then by analytic continuation follow the function $x(\lambda)$ across C; and for every λ in the circle of convergence of each continuation, $x(\lambda)$ will, by the principle of permanence, be a root of (2).

The preceding arguments apply to these new circles of convergence, so that we can extend the function $x(\lambda)$ throughout the entire (finite) λ-plane, with the exception of a *finite number* of singular points. And at each non-singular point λ, $x(\lambda)$ is a root of (2). But also at each singular point we have a root.* The theorem is thus established.

3. *Remark.* A previous argument can be used to show that in every bounded λ-region, $x(\lambda)$ is bounded. This is true in particular in the neighborhood of the singularities of $x(\lambda)$. In consequence, *the singularities† of* $x(\lambda)$ *are branch points at which the function remains finite.* For, the only other possibilities are that a singularity is either a pole or an essential singularity (whether or not also a branch point); but in either case there would exist an infinite sequence of points λ approaching the singularity, for which $|x(\lambda)|$ would approach infinity, and this contradicts the boundedness of $x(\lambda)$.

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* The argument used for $\lambda=\lambda'$ on C applies.

† That is, in the finite plane.

AN APPLICATION OF TENSOR ANALYSIS TO THE FIRST VARIATION OF AN INTEGRAL*

BY J. H. TAYLOR

The method which J. L. Synge† has used to develop the first and second variations of an integral of a restricted type is readily applicable so far as the first variation is concerned, to a general integral of a *regular* calculus of variations problem. Let C be a curve in an n-dimensional space

$$(1) \qquad x^\alpha = x^\alpha(u), \qquad (u_1 \leqq u \leqq u_2), \qquad (\alpha = 1,2, \cdots, n).$$

The value of the integral

$$(2) \qquad t = \int_{u_1}^{u_2} F(x,\dot{x})du$$

will be called the *arc length* of the curve.‡ Let the curve C be imbedded in a one-parameter family of comparison curves

$$(3) \qquad x^\alpha = x^\alpha(u,v), \qquad (u_1 \leqq u \leqq u_2, v_1 \leqq v \leqq v_2),$$

such that C is given by $v=0$, that is, $x^\alpha(u, 0)=x^\alpha(u)$ identically in u. We suppose the limits of integration to be unvaried and so the value of the integral t taken along any one of the curves $v=$constant is a function of v alone. The equations (3) define a surface in terms of the parameters u and v. Let the tangent vectors to the u and v-curves be designated by ξ and η respectively,

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† *The first and second variations of the length-integral in Riemannian space*, Proceedings of the London Mathematical Society, vol. 25 (1925), pp. 247–264.

‡ For the conditions imposed on the function F and the notation here employed see the author's paper, *Parallelism and transversality in a subspace of a general (Finsler) space*, Annals of Mathematics, vol. 28 (1927), pp. 620–628.

$$\xi^\alpha = \frac{\partial x^\alpha}{\partial u}, \qquad \eta^\alpha = \frac{\partial x^\alpha}{\partial v}. \tag{4}$$

We take as corresponding points on the curve C and its comparison curves the points lying on the curves $u = \text{const.}$ The vector η will be called the *variation* or *displacement* vector. Since the limits of integration are independent of v, we have

$$\frac{\partial I}{\partial v} = \int_{u_1}^{u_2} \frac{\partial F}{\partial v} \cdot du$$

the arguments of F being x and ξ.* On account of the homogeneity property of F we may write $F = F_\alpha \xi^\alpha$, where α is to be summed from 1 to n, and where $F_\alpha = \partial F / \partial \dot{x}^\alpha$. Hence we have $\partial F/\partial v = \theta F = \theta(F_\alpha \xi^\alpha)$, where the differential operator θ is defined below. Let a symbol $V^\alpha_\mu(x, \xi)$ be introduced by

$$V^\alpha_\mu(x,\xi) \equiv \Gamma^\alpha_{\mu\beta}\xi^\beta + \frac{1}{2}\frac{\partial \xi^\gamma}{\partial u} f_{\mu\beta\gamma} f^{\beta\alpha}, \tag{5}$$

where the arguments of $\Gamma^\alpha_{\mu\beta}$, $f_{\mu\beta\gamma}$ and $f^{\beta\alpha}$ are x, ξ.† Then the θ-process with respect to the curve C when applied to a contravariant vector X^α is defined by‡

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* Any functions employed will be understood to have the arguments x, ξ unless otherwise indicated.

† The quantities in (5) are defined as follows:

$$f = \frac{1}{2}F^2, \qquad f_{\alpha\beta} = \frac{\partial^2 f}{\partial \dot{x}^\alpha \partial \dot{x}^\beta}, \qquad f_{\alpha\beta\gamma} = \frac{\partial f_{\alpha\beta}}{\partial \dot{x}^\gamma},$$

$f^{\beta\alpha}$ is the element of the reciprocal matrix of $f_{\lambda\mu}$ corresponding to the term $f_{\alpha\beta}$; $\Gamma^\alpha_{\mu} = f^{\alpha\sigma}[\mu\beta,\sigma]$ where

$$[\mu\beta,\sigma] = \frac{1}{2}\left(\frac{\partial f_{\mu\sigma}}{\partial x^\beta} + \frac{\partial f_{\beta\sigma}}{\partial x^\mu} - \frac{\partial f_{\mu\beta}}{\partial x^\sigma}\right).$$

‡ Further properties of the operator θ and the resulting theory of parallelism cannot be given here in detail. Reference may be made to J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas*, Transactions of this Society, vol. 27 (1925), pp. 252–259. This differentiation process was developed about the same time by J. L. Synge, *A generalization of the Riemannian line-element*, Transactions of this Society, vol. 27 (1925), pp. 61–67. The θ-process and resulting parallelism are distinct from the corresponding processes as developed by E. Noether, *Invarianten beliebiger Differentialausdrücke*, Göttinger Nachrichten, 1918,

$$\theta X^\alpha \equiv \frac{dX^\alpha}{du} + V^\alpha_\beta(x,\xi)X^\beta. \tag{6}$$

We adopt the notation of Synge in this connection and designate θX with respect to a v-curve by X and with respect to a u-curve by $\overline{X}$, where X is any tensor.

With this notation, we have

$$\frac{\partial F}{\partial v} = \hat{F} = \hat{F}_\alpha \xi^\alpha + F_\alpha \hat{\xi}^\alpha.$$

If we assume that

$$\frac{\partial^2 x^\alpha}{\partial u \partial v} = \frac{\partial^2 x^\alpha}{\partial v \partial u},$$

it follows that $\xi^\alpha - \bar{\eta}^\alpha = V^\alpha_\lambda(x,\eta)\xi^\lambda - V^\alpha_\lambda(x,\xi)\eta^\lambda$. By means of this relation one obtains

$$\frac{\partial F}{\partial v} = F_\alpha \bar{\eta}^\alpha + Q, \tag{7}$$

where Q is an abbreviation for the expression

$$\hat{F}_\alpha \xi^\alpha + F_\alpha(V^\alpha_\lambda(x,\eta)\xi^\lambda - V^\alpha_\lambda(x,\xi)\eta^\lambda).$$

On expanding these terms and making use of the homogeneity condition, it results that Q is identically zero. Hence, we have

$$\delta t = \delta v \int_{u_1}^{u_2} F_\alpha \bar{\eta}^\alpha du, \tag{8}$$

where the arguments of F are x, ξ of the curve C defined by (3) for $v=0$, and where the directional derivative $\bar{\eta}^\alpha = \theta\eta^\alpha$ of the variation vector η is with respect to the curve C.

From (8) we see that the first variation certainly vanishes if $\bar{\eta}^\alpha = 0$ along the curve. However, if this condition holds

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pp. 37–44, and L. Berwald, *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Mathematische Zeitschrift, vol. 25 (1926), pp. 40–73, except when the θ-process is carried out with respect to a curve which satisfies the equations

$$x''^i = -\phi^i(x,x')$$

of Berwald, p. 44.

at every point of C, the original and varied curve cannot have a point in common unless they coincide. For $\bar{\eta}^\alpha = 0$ is the condition for *parallel displacement*, and under such a displacement the length of a vector when measured with respect to the tangent vector to the curve, as is the case here, remains constant.*

By the homogeneity condition, δt may be written in the form

$$\delta t = \delta v \int_{u_1}^{u_2} \frac{f_{\alpha\beta}\xi^\alpha \bar{\eta}^\beta}{F} du.$$

We now change the form of this expression by means of the relation

$$\frac{\partial}{\partial u}[f_{\alpha\beta}\xi^\alpha\eta^\beta] = \theta[f_{\alpha\beta}\xi^\alpha\eta^\beta] = f_{\alpha\beta}\bar{\xi}^\alpha\eta^\beta + f_{\alpha\beta}\xi^\alpha\bar{\eta}^\beta,$$

and the choice of a parameter such that $F=1$ along the curve C. Then an integration gives

$$\delta t = \delta v\left\{ f_{\alpha\beta}\xi^\alpha\eta^\beta \Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} f_{\alpha\beta}\bar{\xi}^\alpha\eta^\beta du \right\},$$

or

$$(9)\quad \delta t = \delta v\left\{ |\eta|_\xi \cos(\xi,\eta)_\xi \Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} |\bar{\xi}|_\xi |\eta|_\xi \cos(\bar{\xi},\eta)_\xi \, du \right\},$$

where $|\eta|_\xi$ denotes the *length* of the vector η, and $\cos(\xi, \eta)_\xi$ is the cosine of the angle between ξ and η measured with respect to the tangent vector ξ. Here $\bar{\xi}^\alpha$ is the *principal normal* of the curve C and $|\bar{\xi}|_\xi$ is the *first curvature* of the curve.† A curve whose first curvature vanishes will be called a *geodesic*. From (9) we have the following well known result.

THEOREM 1. *A necessary and sufficient condition that the first variation of the length integral of a curve C shall be zero for arbitrary variations except that at the end points the varia-*

* J. H. Taylor, Transactions of this Society, loc. cit., p. 259.

† J. H. Taylor, Transactions of this Society, loc. cit., pp. 261, 263.

tions shall vanish or be orthogonal (with respect to the line elements of C) to the curve, is that C shall be a geodesic. The equations of a geodesic referred to the arc length t as parameter are*†

$$\bar{\xi}^\alpha \equiv \frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\lambda\mu}\left(x, \frac{dx}{dt}\right)\frac{dx^\lambda}{dt}\frac{dx^\mu}{dt} = 0, \quad (\alpha = 1, 2, \cdots, n).$$

The quantity $\delta v|\eta|_\xi \cos(\xi, \eta)_\xi$ represents the projection or component of the displacement vector η on the principal normal ξ of the curve C. *Hence for variations which cause the integrated term of* (9) *to vanish at the limits of integration, the first variation of the arc length of a curve depends only on the displacement of the curve in the direction of its principal normal.*‡

Let R be a region of the surface (3) in which the curves v=constant constitute a one-parameter family of geodesics of the n-space. We suppose that through each point of R there passes one, and only one, of the geodesics of the family. Furthermore, let each of the geodesics of the family be cut transversally (orthogonally with respect to the line elements of the geodesics) by a curve $u = u_1$. Now if $u = u_2$ is another curve in R also cutting the geodesics transversally we see from (9) that the arc length of these geodesics between these two transversal curves is a constant. Conversely, let a given distance be measured off on each geodesic from its point of intersection with the transversal $u = u_1$. Then for each point determined in this fashion we have the condition

* The orthogonality here used includes transversality as a special case. See L. Berwald, loc. cit., p. 56.

† These equations in this form were explicitly obtained by Synge, Berwald and the author at about the same time. J. L. Synge, Transactions of this Society, loc. cit., L. Berwald, *Über Parallelübertragung in Räumen mit allgemeiner Maszbestimmung*, Jahresbereicht der Vereinigung, 1925, p. 217. J. H. Taylor, *Reduction of Euler's equations to a canonical form*, this Bulletin, vol. 31 (1925), p. 262. One observes that this method of treating the first variation leads directly to the Euler equations in *normal* form.

‡ See W. Blaschke, *Vorlesungen über Differentialgeometrie*, vol. I, 1921, p. 35, where this result is given for a curve in euclidean 3-space.

$$|\eta|_\xi \cos(\xi,\eta)_\xi|_{u_2}^{u_1} = 0.$$

By hypothesis $|\eta|_\xi \neq 0$, since the region R is simply covered by the geodesics of the family, and hence $\cos(\xi,\eta)_\xi|_{u_2}^{u_1} = 0$, that is, η is orthogonal to the geodesic, the angle being measured with respect to the tangent vector to the geodesic. Thus we have a theorem which is a simple extension of Kneser's transversality theorem (Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 339).

THEOREM 2. *Let a one-parameter family of geodesics of an n-space ($n>2$) simply cover a region R of a two-dimensional space, and let C_1 be a curve which cuts each of these geodesics transversally (orthogonally with respect to the tangent vector of the geodesic). Then a necessary and sufficient condition that the arc length in the sense of* (2) *measured along these geodesics from the curve C_1 to a second curve C_2 shall be a constant is that C_2 shall also cut the geodesics of the family transversally.*

Let K be a sub-space of k dimensions ($k<n$). A vector ξ at a point of K will be said to be orthogonal to K if it is orthogonal to every vector in K at that point. A necessary and sufficient condition for this is that ξ shall be orthogonal to k linearly independent vectors in K at that point, say the tangent vectors to the parametric curves. Hence if we consider the first variation of an integral with respect to a class of curves joining a fixed point with a point in a given K-space we have the following result.

THEOREM 3. *A necessary and sufficient condition that the first variation of an integral taken along a curve C in an n-space, one end point of which is fixed while the other end point is required to lie in a given k-dimensional space K, ($k<n$), shall vanish is that the curve C shall be a geodesic and that it meet the space K orthogonally with respect to the tangent vector of C.*

THE UNIVERSITY OF WISCONSIN

ON THE MAXIMUM DEFICIENCY OF r-SPACE CURVES

BY B. C. WONG

The problem of determining the greatest deficiency of an r-space curve of given order seems yet unsolved. This problem is equivalent to that of determining the maximum dimension of a sub-space of S_r in which a curve of given order and genus can lie. Thus the maximum genus of a 4-space sextic curve is 2; but if the sextic curve is of genus 3 or 4, it is necessarily a 3-space curve or a plane curve; and if it is of genus greater than 4, it is a plane curve.

It is well known that the deficiency of an r-space curve C^n of order r is zero. Veronese* has shown that the greatest deficiency of an r-space curve C^{r+s} where $s<r$ is s and that of C^{2r} is $r+1$. It is the purpose of this paper to present a method of calculating the maximum deficiency of curves whose orders are greater than $2r$.

To determine the greatest deficiency P of a curve C^n, it is only necessary to determine the least number, h, of apparent double points of its projection on an S_3, for

$$p = (n-1)(n-2)/2 - h. \tag{1}$$

We shall use the phrase *the apparent double points of* C^n instead of *the apparent double points of the 3-space projection of* C^n. To determine h, we make use of the formula†

$$D = \mu_1\mu_2\cdots\mu_{r-1}(\mu_1\mu_2\cdots\mu_{r-1} - \sum\mu_i + r - 2)/2), \tag{2}$$

for the number D of apparent double points of C^n which is the complete intersection of $r-1$ hypersurfaces of order

* *Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens*, Mathematische Annalen, vol. 19 (1882), pp. 161–234.

† Loc. cit., p. 205.

$\mu_1, \mu_2, \cdots, \mu_{r-1}$. D will be a minimum if $r-2$ of the hypersurfaces are hyperquadrics. Hence, putting

$$\mu_1 = \mu_2 = \cdots = \mu_{r-2} = 2, \quad \mu_{r-1} = n/2^{r-2},$$

in (2), we have

$$h \geqq \frac{n}{2^{r-1}}[(2^{r-2} - 1)n - 2^{r-2}(r-2)], \tag{3}$$

and

$$p \leqq \frac{1}{2^{r-1}}[n^2 + 2^{r-2}(r-5)n + 2^{r-1}]. \tag{4}$$

The equality signs hold when $n = 2^{r-2}\,\mu_{r-1}$, where $\mu_{r-1} > 1$. When $n \neq 2^{r-2}\mu_{r-1}$, (3) gives too low a limit for h and (4) gives too high a limit for P. To find the exact value of h we distinguish two cases: n even and n odd.

First take the case n even and equal to $2m$. Consider an equivalent degenerate curve C^{2m} made up of two $(r-1)$-space curves C^m and C'^m lying in different $(r-1)$-spaces but having m points in common. If h' is the least number of apparent double points on each of C^m and C'^m, then

$$h = m^2 - m + 2h',$$

since $m^2 - m$ is the number of apparent intersections of C^m and C'^m. Replacing m by $n/2$ in the above, we have

$$h = n(n-2)/4 + 2h' \tag{5}$$

and

$$p = (n-2)^2/4 - 2h'. \tag{6}$$

Now take the case n odd and equal to $2m-1$. Consider an equivalent degenerate curve C^{2m-1} composed of two $(r-1)$-space curves C^m and C^{m-1} lying in different $(r-1)$-spaces but having $m-1$ points in common. Letting h'' be the least number of apparent double points on C^m and h''' be that of apparent double points on C^{m-1}, we have

$$h = (m-1)^2 + h'' + h''',$$

since $(m-1)^2$ is the number of apparent intersections of C^m and C^{m-1}. If we replace m by $(n+1)/2$, the result is

$$h = (n-1)^2/4 + h'' + h''' \tag{7}$$

and hence

$$P = (n-1)(n-3)/4 - h'' - h'''. \tag{8}$$

To find h' or h'' and h''' we repeat the process. We soon arrive at the component curves whose orders are equal to or less than twice the dimensions of the several sub-spaces in which the component curves lie, and then apply Veronese's rule. Or, in case n is very large, we arrive at the component curves which are all plane and have no double points.

It is to be noticed that $h'=h''=h'''=0$ if $r=3$. Hence the greatest deficiency of any 3-space curve C^n is

$$p = (n-2)^2/4, \qquad \text{for } n \text{ even};$$

and

$$p = (n-1)(n-3)/4, \qquad \text{for } n \text{ odd}.$$

THE UNIVERSITY OF CALIFORNIA

A NOTE ON CERTAIN CONTINUOUS NON-DIFFERENTIABLE FUNCTIONS*

BY F. W. PERKINS

This note gives a treatment of some phases of the theory of a class of functions of which a particular example has already been studied by the author in a note entitled *An elementary example of a continuous non-differentiable function*, in the American Mathematical Monthly (vol. 34 (1927), pp. 476–478). The method there used for the construction of a function with the desired properties bears some resemblance to that used by Brodén, Köpke, and Steinitz for the

* Presented to the Society, September 6, 1928.

construction and study of functions with notable properties with respect to differentiability.* The example consists of a function belonging to a class investigated by Steinitz, but the discussion is somewhat more elementary in form than that given previously in the more general treatment. The method consists, briefly, in assigning functional values at an everywhere dense set of points in such a way that the function can be extended so as to be continuous on the whole interval and to have no derivative. The principle involved enables us not only to construct non-differentiable functions but also to give in simple form a sufficient condition that a continuous function be convertible into a non-differentiable continuous function by a certain change of variable.

First let us examine a few of the more immediate consequences of non-differentiability. We note that such a function cannot be of bounded variation on any subinterval, for by a well known theorem, a function of bounded variation has a derivative almost everywhere, that is, except possibly at the points of a set of measure zero in the sense of Lebesgue.† It follows that a function which has no derivative must have maxima and minima‡ on every subinterval.

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* E. Steinitz, *Stetigkeit und Differentialquotient*, Mathematische Annalen, vol. 52 (1899), pp. 58–69. For a description of the methods of Brodén and Köpke, see Hobson, *The Theory of Functions of a Real Variable*, 3d ed., 1927, vol. 1, pp. 387 ff. See also a paper by E. H. Moore, Transactions of this Society, vol. 1 (1900), pp. 81 ff; and one by Hahn, Jahresbericht der Vereinigung, vol. 26 (1918), pp. 281 ff.

† See, for example, Carathéodory, *Vorlesungen über Reelle Funktionen*, 2d ed., 1927, p. 188, Satz 9, and p. 563 ff.

‡ By a maximum value of a function we shall mean a value attained at a point x', not exceeded at any point in the neighborhood of x', and not maintained constantly in any interval having x' as an interior or end point. The corresponding definition for a minimum is of course also used. It may be noted that a function may have an extremum in this sense which is not an extremum in the "strict" sense in which it is required that a maximum value, for example, be actually greater than all other functional values in the neighborhood of the point at which it is attained. Consider, for instance, the behavior at $x=0$ of the function defined below:

$$\begin{cases} f(x)=0 \text{ when } x=0, \\ f(x)=-x^2\sin^2(1/x) \text{ when } 0<|x|<1. \end{cases}$$

This condition may be expressed analytically in the following manner: Let $f(x)$ be any function continuous on the interval I: $a \leqq x \leqq b$, but not constant on any subinterval of I. A necessary and sufficient condition that $f(x)$ have maxima and minima which are everywhere dense on I is that every subinterval $x_0 \leqq x \leqq x_3$ at the ends of which $f(x)$ assumes distinct values contains two interior points x_1 and x_2 such that $x_1 < x_2$ and

$$\frac{f(x_1) - f(x_2)}{f(x_3) - f(x_0)} > 0.$$

For if $f(x)$ has extrema which are everywhere dense we may obviously choose x_1 and x_2 so that the numerator has the same sign as the denominator. On the other hand, if the fraction is positive, the function $f(x)$ has an extremum between x_0 and x_3. In establishing this fact, we may assume, without loss of generality, that the denominator is positive. In the case in which $f(x_0) \leqq f(x_2)$, we see that $f(x)$ has a maximum between x_0 and x_2; if $f(x_0) > f(x_2)$, we infer that $f(x)$ has a minimum between x_1 and x_3. Moreover, on any subinterval at the ends of which $f(x)$ assumes equal values, $f(x)$ must have an extremum at an interior point. It follows, then, that $f(x)$ has maxima and minima which are everywhere dense on I.

We note that this property in regard to extrema is one which is preserved if we replace the variable x by a variable ξ connected with it by the relation $x = \phi(\xi)$, where the function $\phi(\xi)$ is continuous on the interval $a \leqq \xi \leqq b$ and satisfies the conditions $\phi(\xi') < \phi(\xi'')$ if $\xi' < \xi''$, $\phi(a) = a$, $\phi(b) = b$. The property of non-differentiability is obviously not always preserved by such a change of variable. Hence we see that although the condition that the extrema of $f(x)$ be everywhere dense on I is necessary for non-differentiability, it is not sufficient. Moreover there exist functions with extrema which are everywhere dense and which cannot be rendered non-differentiable by such a change of variable.

We propose to show now that a slight modification of the condition given above for everywhere dense extrema leads to a condition which is sufficient to insure non-differentiability after a suitable change of variable.

THEOREM. *If $f(x)$ is continuous on the interval I: $a \leqq x \leqq b$, but is not constant on any subinterval, and if there exists a positive quantity δ such that every subinterval $x_0 \leqq x \leqq x_3$ at the ends of which $f(x)$ assumes distinct values contains two points x_1 and x_2 such that $x_1 < x_2$ and*

$$\frac{f(x_1) - f(x_2)}{f(x_3) - f(x_0)} > \delta,$$

then there exists a function $\phi(\xi)$, continuous and monotonically increasing on the interval $a \leqq \xi \leqq b$, such that $F(\xi) = f[\phi(\xi)]$ fails to have a derivative with respect to ξ at every point of the interval $a \leqq \xi \leqq b$.

We shall first treat the case in which $f(x)$ attains its least and greatest values on I at the points $x = a$ and $x = b$ respectively: $f(a) \leqq f(x) \leqq f(b)$ or $f(b) \leqq f(x) \leqq f(a)$ when $a \leqq x \leqq b$.

We consider first the value of the fraction

$$\frac{f(x_1) - f(x_2)}{f(b) - f(a)}$$

for values of x_1 and x_2 such that $a \leqq x_1 \leqq x_2 \leqq b$. Since this is a particular form of the fraction given in the hypothesis of the theorem, we know that its greatest value must be greater than δ. We note that to attain this maximum value we may have neither $x_1 = a$ nor $x_2 = b$. The fraction may conceivably attain its maximum value at more than one pair of points x_1 and x_2. However, the set of all points which may be used as x_1 is closed, and therefore contains a point farthest to the left. We shall call this point X_1. All the points which are possible choices for an x_2 corresponding to $x_1 = X_1$ form a closed set, and therefore contain a point farthest to the right, which we shall call X_2. We now repeat the same process for each of the intervals $a \leqq x \leqq X_1$,

$X_1 \leqq x \leqq X_2$ and $X_2 \leqq x \leqq b$. It will be noted that the function $f(x)$ attains its extreme values on any one of these intervals at the ends of that interval. The three new pairs of points thus obtained, taken together with the points $x = X_1$ and $x = X_2$, divide the interval I into nine parts, to each of which we apply the process again, and so on indefinitely.

The set $\{X\}$ of all points thus obtained is everywhere dense on I. For if this is not true there exists a subinterval γ containing none of these points. Let μ be the difference between the greatest value and the least value of $f(x)$ on γ. Since $f(x)$ is not constant on any interval we know that $\mu > 0$. Let I_1 be that subinterval obtained by the first subdivision (by the points X_1 and X_2) which contains γ; let I_2 be that subinterval obtained by the second subdivision which contains γ, and so on. In this way we get an infinite sequence of subintervals $I_1, I_2, I_3, \cdots$, each of which contains all the following, and also contains γ. Let $X_1, X_1', X_1'', \cdots$, be the left hand end points of these intervals. Each of these points lies to the left of the next following point of the sequence, and also to the left of γ. Consequently they have a unique limit point, X, on I. Moreover, at any two successive points of the sequence $X_1, X_1', X_1'', \cdots$, the function $f(x)$ has values which, it is readily seen, differ by at least μ. This implies that $f(x)$ is discontinuous at $x = X$, contrary to our hypothesis. This contradiction establishes the desired result. It may be noted, incidentally, that the points of the everywhere dense set which we have just described yield extrema of the function $f(x)$.

We now proceed to the definition of the function $\phi(\xi)$. Since this function is to be continuous it is possible to define it completely by assigning suitable functional values at the points of an everywhere dense set. First we set $\phi(a) = a$, $\phi(b) = b$, and $\phi(\xi_1) = X_1$ and $\phi(\xi_2) = X_2$, where*

* Since $F(\xi) = f[\phi(\xi)]$, while $\phi(a) = a$ and $\phi(b) = b$, it follows that $F(a) = f(a)$, and $F(b) = f(b)$. However in analogous work a little later it will prove necessary to distinguish carefully between the functions F and f.

$$\begin{cases} \xi_1 = \dfrac{aF(b) - bF(a) + f(X_2)[b-a]}{F(b) - F(a)}, \\ \xi_2 = \dfrac{aF(b) - bF(a) + f(X_1)[b-a]}{F(b) - F(a)}. \end{cases}$$

These formulas may be given a simple geometric interpretation: ξ_1 and ξ_2 are the abscissae of the points of intersection of the lines joining $[a, F(a)]$ and $[b, F(b)]$ and the lines $\eta = f(X_2)$ and $\eta = f(X_1)$, respectively, in the ξ, η plane.

Each of the subintervals into which X_1 and X_2 divide I contains two points standing in the same relation to that subinterval as X_1 and X_2 do to I. By the method given above we determine the corresponding values of ξ. We repeat this process indefinitely, obtaining an infinite set of points at each of which the function $\phi(\xi)$ is defined. The order relations between these points is the same as the order relations between the corresponding points X, and so $\phi(\xi)$, in so far as it is now defined, is a monotonically increasing function.

The straight line joining $[a, F(a)]$ and $[b, F(b)]$ is the graph of a function $F_0(\xi)$ which is a first approximation to the function $F(\xi)$. The broken line joining the points $[a, F(a)]$, $[\xi_1, F(\xi_1)]$, $[\xi_2, F(\xi_2)]$ and $[b, F(b)]$ yields a second approximation, $F_1(\xi)$; a third approximation may be obtained by replacing each segment of this line by a broken line of three pieces related to the segment in the same way as the graph of the second approximation is to that of the first, and so on.*

Denoting by λ_i the slope of a line segment of the graph of an approximating function $F_i(\xi)$, we see, by a simple computation, that the slope of the second of the line segments by which it is replaced in the graph of $F_{i+1}(\xi)$ is $-\lambda_i$, while the slope of each of the other two segments is at least as great numerically as $|\lambda_i|(1+\delta)$. This enables us to prove that the

* The sequence $F_0(\xi)$, $F_1(\xi)$, $F_2(\xi)$, $\cdots$ converges uniformly, but this is of no importance for our proof, as we shall obtain $F(\xi)$ by another method. We shall have occasion, however, to use these functions in another connection.

point set on which we have already defined the function $\phi(\xi)$ is everywhere dense. For if such is not the case there exists an interval throughout which each approximating function is linear with a slope at least as great numerically as the positive quantity $|\lambda_0|$. Consequently there exists a number $c>0$ and pairs of points of the set $\{X\}$, as near together as we choose to stipulate in advance, at which the function $f(x)$ assumes values differing by at least c. But this contradicts the hypothesis of continuity, and so shows that $\phi(\xi)$ is defined on an everywhere dense set of points.

At any point $\xi=\xi'$ at which $\phi(\xi)$ is not already defined we set $\phi(\xi)$ equal to the least upper bound of all its known values on the interval $a\leqq\xi<\xi'$. The function $\phi(\xi)$ is now defined and monotonically increasing on the interval $a\leqq\xi\leqq b$. It is continuous there; for a discontinuity could only be a finite jump, which would imply the existence of a subinterval of I free from points of the set $\{X\}$, contrary to the known fact that this set is everywhere dense.

We shall now show that the continuous function $F(\xi)=f[\phi(\xi)]$ has no derivative at any point of the interval $a\leqq\xi\leqq b$. We understand this statement to mean that at no point $\xi=\xi'$ of this interval does the difference quotient $[F(\xi'+h)-F(\xi')]/h$ approach a limit as h approaches zero.*

If ξ' corresponds to one of the points of the set $\{X\}$, then it is clear that $F(\xi)$ has no derivative at $\xi=\xi'$, because, for sufficiently large values of n, the forward derivative of $F_{n+1}(\xi)$ at that point is at least $1+\delta$ times as great numerically as the forward derivative of $F_n(\xi)$ at $\xi=\xi'$, and so the forward derivative of $F_{n+k}(\xi)$ at $\xi=\xi'$ can be made as large numerically as we like by choosing k sufficiently large. Since the forward derivatives of $F_n(\xi)$, $F_{n+1}(\xi)$, $F_{n+2}(\xi)$, $\cdots$ at $\xi=\xi'$ give us the values of the difference quotient $[F(\xi'+h)-F(\xi')]/h$ for a sequence of values of h approaching zero as a limit, we see that the difference

* Some writers use a definition of differentiability which allows the difference quotient either to approach a limit or to become positively or negatively infinite.

quotient cannot approach a limit in this case. Backward derivatives of the approximating functions may be used equally well. The forward and backward derivatives have opposite signs, and so we see that for a certain sequence of positive values of h tending toward zero the difference quotient becomes infinite with one sign, while for a certain sequence of negative values of h tending toward zero the difference quotient becomes infinite with the other sign. It should be noted, however, that this does not necessarily mean in either case that the difference quotient becomes infinite independently of the manner in which h approaches zero from one side.*

It may also be noted that $F(\xi)$ has no forward derivative at the point $x=a$ and no backward derivative at $x=b$, though the difference quotient may conceivably become positively or negatively infinite at these points.

It remains to show that $F(\xi)$ has no derivative at any point ξ' of the open interval $a<\xi<b$ which does not correspond to a point of the set $\{X\}$. At such a point each of the functions $F_n(\xi)$ has a derivative. Let $\xi=\xi_n''<\xi'$ and $\xi=\xi_n'''>\xi'$ be points corresponding to points of the set $\{X\}$ between which $F_n(\xi)$ has a constant derivative. If we set $h=h_n''=\xi_n''-\xi'$ or $h=h_n'''=\xi_n'''-\xi'$, the value obtained for the difference quotient $[F(\xi'+h)-F(\xi')]/h$ is in one case not greater algebraically than $D_\xi F_n(\xi')$ and in the other case not less algebraically than $D_\xi F_n(\xi')$. Since $\lim h_n''=\lim h_n'''=0$, it

* In special cases, however, this may happen. In the case of the function previously studied by the author, cited above, it can readily be shown that at each of the everywhere dense set of maxima (given by $x=p/3^n$, where n is any positive integer and p any odd positive integer less than 3^n) the difference quotient becomes positively infinite on the left and negatively infinite on the right, while at the minima (given by the remaining values of x of the form $p/3^n$ in the open interval $0<x<1$) the difference quotient becomes negatively infinite on the left and positively infinite on the right. For $x=0$ and $x=1$ the difference quotient becomes positively infinite. The coordinates of the extrema are here given for the function as defined in the earlier paper. It is readily verified that the properties stated regarding the behavior of the difference quotient are preserved (in this case) by the transformation described in the present paper.

follows that this difference quotient can approach a limit only if $D_\xi F_n(\xi')$ approaches a limit as n becomes infinite. This, however, is never the case. In the ratio $D_\xi F_{n+1}(\xi')/D_\xi F_n(\xi')$ neither the numerator nor the denominator is zero and neither approaches zero as a limit. Moreover for any n this ratio has either the value -1 or else is greater than $1+\delta$, and so cannot approach the limit unity, as it would if $D_\xi F_n(\xi')$ approached a limit. Hence the difference quotient $[F(\xi'+h)-F(\xi')]/h$ cannot approach a limit.

This completes the proof of the theorem for the case in which either $f(a) \leqq f(x) \leqq f(b)$ or $f(b) \leqq f(x) \leqq f(a)$. The proof can be extended to the general case immediately by dividing the interval I into a finite or denumerably infinite set of subintervals, each of which satisfies the above condition, and then applying the above reasoning to each such subinterval. This subdivision of I can be accomplished by choosing as the first subinterval the interval between points at which $f(x)$ attains its extreme values. In case these values are attained at more than one pair of points we may choose one particular pair by means of the device used in the similar situation for the choice of X_1 and X_2. We then apply this method to the remaining interval or intervals and continue the process indefinitely, or until we have exhausted I.

DARTMOUTH COLLEGE

A NEW PROOF OF CERTAIN RELATIONS OF MORSE IN THE CALCULUS OF VARIATIONS IN THE LARGE*

BY D. E. RICHMOND

1. *Introduction.* In a recent paper,† M. Morse has developed a theory of calculus of variations "in the large," in which he has made use of the results of his paper‡ on critical points of functions of n variables. He establishes a parallelism between types of critical points and types of extremal arcs in which the type of an extremal arc whose end points are not conjugate is completely determined by the number of mutual conjugate points within the arc. As a first application of the critical point theory, he obtains a set of necessary relations between the numbers of extremal arcs of different type, which join two given points. He also discusses periodic extremals as to type, and obtains number relations from the critical point theory.

Morse's relations are obtained for the parametric problem under certain general boundary conditions and under the hypothesis of the existence of a field of extremals covering the region of operation.

The present paper treats only the number relations between extremal arcs which join two points. Using Morse's hypotheses, we derive these relations rather simply from well known continuity properties of the solutions of differential equations, without making use of the critical point theory or n-dimensional analysis situs. It should be stated, however, that Morse has since proved the relations without

* Presented to the Society, September 7, 1928.

† M. Morse, *The foundations of a theory in the calculus of variations in the large*, Transactions of this Society, vol. 30 (1928), pp. 213–274.

‡ M. Morse, *Relations between the critical points of a real function of n independent variables*, Transactions of this Society, vol. 27 (1925), pp. 345–396.

the field hypothesis* and has also announced results for the n-dimensional calculus of variations problem. It seems improbable that our method can be generalized to treat these problems.

The method here presented is of interest, however, because in this case, it makes possible the statement of the relations in a somewhat more restrictive form than that given by Morse, in fact, in a form which immediately gives a new existence theorem. Moreover, our method gives an insight into the relative positions of the extremal arcs of different type, and is thus supplementary to the critical point method.

2. *Hypotheses and Relations of Morse. Let the given integral be*

$$\int_{t_0}^{t_1} F(x, y, \dot{x}, \dot{y})dt. \tag{1}$$

I. *We assume that the function $F(x, y, \dot{x}, \dot{y})$ is positively homogeneous of the first degree in $\dot{x}$ and $\dot{y}$, and analytic in all its arguments for (x, y) any point interior to an open region S of the (x, y) plane and $\dot{x}$ and $\dot{y}$ any two numbers not both zero. We also assume that for the same arguments, $F_1(x, y, \dot{x}, \dot{y}) > 0$.*

II. *Let there be given a closed region S_1, interior to S, bounded by a simple closed curve β consisting of a finite number of analytic arcs. We assume that β is extremal-convex in the following sense*: (1) *The interior angles at the vertices of β all lie between* 0 *and* π; (2) *An extremal tangent to any arc of β lies outside of S_1 in the neighborhood of the point of contact.*

III. *We assume that S_1 is covered in a one-to-one manner by a proper field of extremals of the form*

$$x = h(u, v), \qquad y = k(u, v), \tag{2}$$

where u is the parameter, and v is the arc length measured along the extremals, and where at every point (u, v) that corresponds to a point (x, y) in S_1, the functions (2) *are single-valued and analytic in u and v, and*

* See Morse, first paper, loc. cit., footnote, p. 223.

$$D = \begin{vmatrix} h_u & h_v \\ k_u & k_v \end{vmatrix} \neq 0.$$

IV. *We assume that the problem is reversible.**

The region S_1 in the (x, y) plane corresponds to a closed, extremal-convex region R in the (u, v) plane, whose boundary γ is a simple closed curve consisting of a finite number of analytic arcs. If $\dot{u} > 0$ and we let $t = u$, the integral (1) can be put into non-parametric form with an integrand, $f(u, v, v')$, where $v' = dv/du$. The non-parametric problem is regular ($f_{v'v'} > 0$). The field extremals become the lines $u =$ constant, which of course are not included among the solutions of the non-parametric problem. All other extremals of the parametric form are included in the non-parametric problem because of the assumption of reversibility.

Morse proves a lemma (p. 224) which we take over. It amounts to the statement that in the simply-connected region R, no line $u =$ constant cuts the boundary γ in more than two points.

In R, to the extremals $u =$ constant, no other extremals can be tangent. Hence, any extremal arc, other than $u =$ constant, is representable in the form $v = M(u)$, where $M(u)$ is analytic for that part of the arc which lies within R.

Let A and B be any two points, within or on the boundary of R, and not on the same line $u =$ constant. We state two preliminary results: (a) *There exists at least one extremal arc which joins A to B in R and which has no points on the boundary of R, with the possible exception of A and B*; (b) *The points A and B can be joined by at most a finite number of extremal arcs in R.* We shall prove (a) in §3, while (b) is proved in Morse's paper (p. 226) without the use of the critical point theory.

We consider the set G of arcs which join A to B in R. For the present, we assume that there exists no arc in G upon which B is conjugate to A. Each arc in G has a type number

* See Morse, loc. cit., p. 223.

k which will be defined to be equal to the number of points on the arc conjugate to A. Let m be the maximum of the type numbers k and $M_k(k=1, 2, \cdots, m)$ be the number of extremal arcs in G of type k. The relations of Morse are then the following:

$$(3)\quad \begin{cases} M_0 \geqq 1, \\ M_0 - M_1 \leqq 1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ (-1)^{m-1}[M_0 - M_1 + \cdots + (-1)^{m-1}M_{m-1}] \geqq 1, \\ M_0 - M_1 + \cdots + (-1)^m M_m = 1. \end{cases}$$

3. *A Preliminary Theorem.* Morse gives the following consequence of the assumptions of §2. *There exists a positive constant e so small that any two points in R whose abscissae differ in absolute value by less than e, can be joined by a unique extremal arc, all of whose points (except possibly the end points) lie within R.*

We take over this result and show that it follows that, if A and B are any two points in R, not on the same line $u=$constant, there exists at least one extremal arc E which joins A to B in R, and that no points of E, except possibly A and B, lie on the boundary γ of R. Let A and B be represented by (u_0, v_0) and (u_1, v_1), respectively, $u_0 < u_1$. By virtue of the result just stated, a finite number p of lines of the form $u=$constant can be chosen between $u=u_0$ and $u=u_1$ such that, if p arbitrary points are chosen in R, one on each of these lines, there is determined a unique chain of extremal arcs, which joins A to B. The value of the fundamental integral taken along such chains becomes a single-valued continuous function of the p ordinates. The domain of definition is closed in each of the variables. Hence there exists a set of ordinates for which the integral takes on its minimum value. The extremal arcs of the corresponding chain join on to each other so as to form a single extremal arc E from A to B, since in a regular problem a curve with corners cannot minimize the integral. Now E has no points

on the boundary γ with the possible exception of A and B. This statement follows from the extremal-convex boundary and from the impossibility of corners.

We now prove the following theorem.

THEOREM 1. *If there exists in R, joining A to B, an extremal arc E_0 of type $n>0$, there exist in R, joining A to B, at least two other extremal arcs, one of which lies above E_0, the other below E_0.*

Proof. Let E_0 be extended in both senses until it intersects γ, the boundary of R, thus dividing R into two regions, R_1 and R_2. Now R_1 is extremal-convex in a somewhat broader sense than that of §2 II, in that part of the boundary is itself an extremal arc. But as before, A and B can be joined in R_1 by an extremal arc E which minimizes the integral in comparison with all other arcs which join A to B in R_1. The points of E lie within R_1 or on its boundary. But E cannot coincide with E_0 because the latter contains at least one point conjugate to A by hypothesis and hence cannot minimize the integral. In fact, it is readily proved that E has no points except A and B in common with the boundary of R_1. A similar argument applies to R_2.

4. *The Extremals through A.* Let A and B be represented by (u_0, v_0) and (u_1, v_1), respectively. Let E_0 be an extremal arc in G of type $n \geqq 0$. We consider the family of extremals through A. Any such extremal E is defined by the angle α, measured from E_0 to E at A, since in a regular problem there exists* one and only one extremal through a given point with a given slope. Hence the family may be represented by

$$v = v(u, \alpha). \tag{4}$$

Suppose that a particular extremal E of (4) makes an angle of $\alpha_0 \neq 0$ with E_0 at A, and intersects E_0 exactly r times between A and B. The abscissae of these points of

* Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 184–185.

intersection will be denoted by $U_i (i = 1, 2, \cdots, r)$, where for each i, $u_0 < U_i < u_1$. The number of intersections r will depend upon α_0. An arc AP_i of E has no points on the boundary of R, except possibly A. This follows from the regularity of the problem (extremals have no corners) and from the extremal-convex nature of the boundary. For α sufficiently near α_0 and $u_0 < u \leqq U_i$, the extremals (4) lie entirely within R, since $v(u, \alpha)$ is a continuous function* of α. Let E_0 be represented by $v = f(u)$, $u_0 \leqq u \leqq u_1$. Then at P_i we have

$$v(U_i, \alpha_0) = f(U_i), \quad (i = 1, 2, \cdots, r).$$

Since* $v(u, \alpha)$ and $f(u)$ are of class C' and $v_u(U_i, \alpha_0) \neq f'(U_i)$ for $\alpha_0 \neq 0$, the implicit function theorem is applicable and for α near α_0 the abscissa U_i of the ith point of intersection of the extremals (4) with E_0 is a continuous differentiable function of α. In fact, for α near α_0, we have identically

$$v[U_i(\alpha), \alpha] = f[U_i(\alpha)], \quad (i = 1, 2, \cdots, r).$$

Differentiating with respect to α, we have

$$v_u[U_i(\alpha), \alpha] \cdot U_i'(\alpha) + v_\alpha[U_i(\alpha), \alpha] = f'[U_i(\alpha)] \cdot U_i'(\alpha).$$

Rearranging and omitting arguments, we obtain

$$v_\alpha = U_i'(f' - v_u). \tag{5}$$

Here f' and v_u represent the slopes of E_0 and E respectively, at their ith point of intersection.

By definition, any function $U_i(\alpha)$ will be restricted by the inequality $u_0 < U_i(\alpha) \leqq u_1$.

For α sufficiently small, exactly n functions $U_i(\alpha)$ exist (that is, $r = n$) and the abscissae of the conjugate points on E_0 are given† by $U_i(0)$, $(i = 1, 2, \cdots, n)$. For numerically larger values of α, some of these functions may cease to be defined, or on the other hand, functions $U_i(\alpha)$ may exist for $i > n$.

If $U_k(\alpha_0) < u_1$, we have seen that the function $U_k(\alpha)$ is defined for α sufficiently near α_0. The ends of the intervals

* Bolza, loc. cit., p. 73.

† Bolza, loc. cit., p. 79.

of definition of $U_k(\alpha)$ must therefore correspond to values of α for which $U_k(\alpha)=u_1$.

Since the problem is regular, no extremal is tangent to any other in R. The points of intersection of two extremals are therefore discrete. Hence if $U_k(\alpha)$ is defined on a certain α interval, the functions $U_i(\alpha)$, $i<k$, are also defined on this interval and

$$U_i(\alpha) < U_j(\alpha), \quad i < j \leqq k.$$

The necessary and sufficient condition that an extremal E, through A, pass through B is that

$$U_k(\alpha) = u_1, \tag{6}$$

for some integer k. For values of α for which (6) holds, $U_k(\alpha)$ will have in general only a one-sided derivative.

In discussing equation (5), with i replaced by k, it will suffice for our purposes to consider the case $\alpha>0$. Then, if k is odd, $v_u<f'$ and v_α takes the sign of U_k'. If k is even, $v_u>f'$ so that v_α and U_k' have opposite signs.

Now $v_\alpha(0, \alpha)=0$ and $v_{\alpha u}(0, \alpha)>0$. Hence the conjugate points to A on E are given* by those values of u for which $v_\alpha(u, \alpha)=0$. Then if $v_\alpha>0$ for $u=u_1$, there exist an even number of points conjugate to A between A and B; if $v_\alpha<0$, an odd number. From (5) therefore, we obtain the following theorem.

THEOREM 2. *Let the functions $U_i(\alpha)$ be defined relative to an extremal arc E_0, which joins A to B. Suppose E is an extremal arc which joins A to B and which makes an angle $\alpha_0>0$ with E_0 at A. Let n be the type of E and k the integer such that $U_k(\alpha_0)=u_1$, the abscissa of B. Then n is odd if $U_k'(\alpha_0)>0$ and k is even, or if $U_k'(\alpha_0)<0$ and k is odd; n is even if $U_k'(\alpha_0)>0$ and k is odd, or if $U_k'(\alpha_0)<0$ and k is even. Finally, B is conjugate to A if $U_k'(\alpha_0)=0$, for any k.*

5. *Types of Adjacent Arcs.* Now let the extremal arcs of G be arranged in order of their slopes at A. Two arcs

* Bolza, loc. cit., p. 78.

which are consecutive in this order will be said to be adjacent to each other. We now prove the following theorem.

THEOREM 3. *Adjacent extremal arcs in G differ in type by unity.*

Let E and $\overline{E}$ be the adjacent extremal arcs, E being that arc which has the lesser slope at A. Let n and $\bar{n}$ be the type of E and $\overline{E}$ respectively. Suppose first that the functions $U_i(\alpha)$ are defined relative to E.

Since E is of type n, we know that $U_n(0) < u_1$ and that $U_{n+1}(\alpha)$ is not defined for α sufficiently small. Let α_0 be the angle at A from E to $\overline{E}$. Then $U_k(\alpha_0) = u_1$ for some integer k, which it remains to determine. Now $U_n(\alpha_0)$ is defined. Otherwise we would have $U_n(\alpha) = u_1$ for some positve $\alpha < \alpha_0$, which would imply that $\overline{E}$ was not adjacent to E. There are therefore two cases:

(A) $$U_n(\alpha_0) = u_1,$$

(B) $$U_n(\alpha_0) < u_1,$$

CASE A. $U_n(\alpha_0) = u_1$. We have assumed that B is not conjugate to A on $\overline{E}$. Then by Theorem 2, $U_n'(\alpha_0) \neq 0$. Since $U_n(\alpha) < u_1$ for $0 \leqq \alpha < \alpha_0$, it now follows that $U_n'(\alpha_0) > 0$.

CASE B. $U_n(\alpha_0) < u_1$. Here $U_{n+1}(\alpha_0)$ must exist. If $U_{n+1}(\alpha_0) < u_1$, there exists a positive $\alpha < \alpha_0$ for which $U_{n+1}(\alpha) = u_1$, which implies that $\overline{E}$ and E are not adjacent. Hence $U_{n+1}(\alpha_0) = u_1$. Since $U_{n+1}'(\alpha_0) \neq 0$ and $U_{n+1}(\alpha)$ is not defined for $0 \leqq \alpha < \alpha_0$, it follows that $U_{n+1}'(\alpha_0) < 0$. Thus we must have one of the following possibilities:

(A) $$U_n(\alpha_0) = u_1, \qquad U_n'(\alpha_0) > 0;$$

(B) $$U_{n+1}(\alpha_0) = u_1, \qquad U'_{n+1}(\alpha_0) < 0.$$

Using Theorem 2, with $n = \bar{n}$ and $k = n$ and $n+1$ respectively, we now see that, for either (A) or (B), if n is odd, $\bar{n}$ is even; while if n is even, $\bar{n}$ is odd. Hence in particular, $n \neq \bar{n}$.

We now reverse the roles of E and $\overline{E}$. Let the functions

$\overline{U}_i(\alpha)$ be defined relative to $\overline{E}$ in the same manner that the functions $U_i(\alpha)$ were defined relative to E.

For Case A, we have

$$\overline{U}_n(-\alpha_0) = U_n(\alpha_0) = u_1,$$

which implies that

$$\overline{U}_i(-\alpha_0) < u_1, \qquad (i < n).$$

Since the functions $\overline{U}_i(\alpha)$ are continuous and bounded, it follows that

$$\overline{U}_i(0) < u_1, \qquad (i < n).$$

Perhaps also $\overline{U}_n(0) < u_1$, but certainly $\overline{U}_{n+1}(0)$ is not defined since E is adjacent to $\overline{E}$. Since the numbers $\overline{U}_i(0)$ are the abscissae of the conjugate points to A on $\overline{E}$, the type $\bar{n}$ of $\overline{E}$ is either n or $n-1$. But we proved that $\bar{n} \neq n$, so that for Case A there remains the single possibility $\bar{n} = n-1$.

A similar argument shows that for Case B, $\bar{n} = n+1$. Combining the results of both cases, $\bar{n} = n-1$ or $n+1$, as the theorem states.

6. *Proof of Number Relations.* Let the extremal arcs which join A to B in R, be arranged in order of their slopes at A. The first and last must be of type zero. Otherwise, according to Theorem 1, there would exist extremal arcs of G whose initial slopes were respectively less than and greater than the initial slopes of these arcs.

Since adjacent arcs differ in type by unity, we see that in order that the first and last shall be of type zero, the total number of extremal arcs in G must be odd. Let us remove from G either the first or last arc and group the remaining ones into adjacent pairs. If m is the maximum type number which occurs in G, the integers M_k must be of the following form:

$$(7) \qquad \begin{cases} M_0 = 1 + p_1, \\ M_1 = p_1 + p_2, \\ \cdots \quad p_i \geqq 1, (i = 1, 2, \cdots, m), \\ M_{m-1} = p_{m-1} + p_m, \\ M_m = p_m, \end{cases}$$

where the integers p_i represent the total number of pairs, which comprise extremal arcs of types i and $i-1$. It is readily seen that there exists at least one pair for each value of i.

By addition and subtraction of the equations of (7), we obtain the relations (3) of Morse. But a stronger form of these relations follows if we substitute $p_i = q_i + 1$ in (7). Then

$$(7') \quad \begin{cases} M_0 = 2 + q_1, \\ M_1 = 2 + q_1 + q_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \qquad\qquad q_i \geqq 0, (i = 1, 2, \cdots, m), \\ M_{m-1} = 2 + q_{m-1} + q_m, \\ M_m = 1 + q_m. \end{cases}$$

Now by addition and subtraction, we obtain the following theorem.

THEOREM 4. *If M_k represents the number of extremal arcs of type k which join A to B, and if m is the maximum of the types k, then*

$$(8) \quad \begin{cases} M_0 \geqq 2, \\ M_0 - M_1 \leqq 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ M_0 - M_1 + \cdots + (-1)^m M_m = 1, \end{cases}$$

where the next to the last relation is

$$M_0 - M_1 + \cdots + (-1)^{m-1} M_{m-1} \geqq 2 \quad \text{or} \quad \leqq 0,$$

according as m is even or odd.

From (7') we have the following new theorem.

THEOREM 5. *If M_k represents the number of extremal arcs of type k which join A to B and if m is the maximum of the types k, then*

$$(9) \quad \begin{cases} M_i \geqq 2, \quad (i = 1, 2, \cdots, m-1), \\ M_m \geqq 1. \end{cases}$$

We have as an immediate corollary,

COROLLARY. *If there exists in G an extremal arc of type m, there exist in G at least 2m other extremal arcs.*

It is of interest to consider the geodesics, on a torus generated by revolving a circle C of radius a about an axis in its plane which does not intersect it. Let u measure the angle through which C has been revolved from an initial position, and let v represent arc length measured along V from the point farthest from the axis. Let $A(u_0, 0)$ and $B(u_1, 0)$ be the end points of a segment of the geodesic $v=0$ of type m. The geodesics which join A to B and lie in the uv plane within the rectangle bounded by $u=u_0$, $u=u_1$, $v=\pi a$, $v=-\pi a$ are then exactly $2m+1$ in number and the integers M_i satisfy the equalities in (9). By choosing the two points sufficiently far apart, m may be made arbitrarily large.

7. *A Restriction Removed.* It is now necessary to remove the restriction so far imposed, namely, that on no arc in G is B conjugate to A. Here we adopt the procedure of Morse (pp. 231–232), merely stating the results. A point B' may be chosen near B and not on an envelope of the extremals through A. For the extremal arcs which join A to B', relations (8) hold. Now allow B' to approach B continuously in such a manner as to avoid envelopes. It will be found *that relations* (8) *continue to hold for* $B=B'$, provided that certain conventions are adopted as to the types of the extremal arcs in G. These conventions are as follows. *Let g be an extremal arc joining A to B on which there are m points conjugate to A prior to B. If B is not conjugate to A, g is to be counted as one arc of type* $k=m$. *If B is conjugate to A and B is an ordinary point on the* $(m+1)$*st envelope T of the extremals through A, g is to be counted as two extremal arcs of types* $k=m+1$ *and* $k=m$, *respectively. If B is conjugate to A and T has a cusp at B which opens toward A, g is to be counted as one extremal arc of type* $k=m+1$. *If T has a cusp at B which opens away from A, g is to be counted as one arc of type* $k=m$.

WILLIAMS COLLEGE

ALEXANDER ZIWET—IN MEMORIAM

In the death on November 18, 1928, of Alexander Ziwet, Professor Emeritus of Mathematics in the College of Engineering of the University of Michigan, this Society lost one of its most distinguished scholars.

Professor Ziwet was born in Breslau, Germany, February 8, 1853, of a Polish father and a German mother. His father was a landed proprietor, and his early education was at home under private tutors. At the age of twelve he entered the gymnasium at Glogau, German Silesia, and after five years attendance was graduated in 1870. His father having died in the meantime, he joined his mother and sisters who had gone to Warsaw, and there he learned Polish and Russian and entered the University. In 1873 he transferred to the University of Moscow, which he left in 1874 to study engineering in Germany. After some time in an architect's office, he entered the Polytechnic School at Karlsruhe, Baden, and in 1880 was graduated as a civil engineer. He had served meanwhile the required year in the army, and had passed his examination for a commission as a lieutenant in the reserve artillery corps.

In the fall of 1880 he came to Detroit, and, after about two years in the Lake Survey and five years in the computing division of the United States Coast and Geodetic Survey, he was appointed an instructor in mathematics at the University of Michigan. After several promotions in rank, he became professor of mathematics and head of the Department of Mathematics in the College of Engineering. He was also head of the Department of Modern Languages in the College of Engineering for several years. He had a command of seven modern languages. He retired in 1925, becoming Professor Emeritus of Mathematics. The University conferred on him the honorary degree of Doctor of Science in 1927.

Professor Ziwet has been prominently connected with the development of mathematics in the United States. He was a member of the editorial board of this Bulletin from 1892 to 1920, and served as vice-president of the American Mathematical Society during 1903. His numerous book reviews showed not only a broad knowledge of mathematics, but also a keen critical sense. He helped in the establishment of the Chicago section of the Society in 1897, was a member of its first program committee, and gave it his unqualified support from the beginning by regular attendance and discussion. He took a keen interest in the Mathematical Congress at Chicago, and in the first colloquium of the Society at Evanston, in the latter to the extent of writing up for publication by the Society the lectures delivered by Klein at that time. He was vice-president and chairman of Section A of the American Association for the Advancement of Science during 1905. His retiring address on *The relation of mechanics to physics* was published in Science, January 12th, 1906. He was also a member of the executive council which organized the Mathematical Association of America in 1916.

To his colleagues and to serious students Professor Ziwet gave most

generously of his time and thought. His familiarity with the literature of mathematics was amazing, covering long periods of time and journals in half a dozen languages. He was one of the founders of the University of Michigan Mathematical Club. For many years the club met at his rooms. To its activities, he devoted much time and attention. He was also active in promoting the general scholarly interests of the University, being one of the influential members of the University of Michigan Research Club.

Professor Ziwet left a library of over 5000 volumes which he presented to the University of Michigan. In the collection there were about 400 volumes on the history of mechanics, including many notable works such as a first edition of Newton's *Principia*. The library owes much to him for constant attention over a long period of years. Professor Ziwet's library, together with Professor Beman's, which was also large, and which was presented to the University at the time of Professor Beman's death, are valuable additions to the Mathematical Library of the University.

The following is a list of Professor Ziwet's publications.

An Elementary Treatise on Theoretical Mechanics, Macmillan, 1894, 591 pp.

Elements of Theoretical Mechanics, Macmillan, 1904, 494 pp.

Introduction to Analytical Mechanics (with P. Field), Macmillan, 1912, 378 pp.

Analytic Geometry and Principles of Algebra (with L. A. Hopkins), Macmillan, 1913, 369 pp.

Analytic Geometry (with L. A. Hopkins), Macmillan, 1916, 280 pp.

Note on the composition of finite rotations about parallel axes, this Bulletin, vol. 14 (1908), pp. 208–212.

A note on plane kinematics (with P. Field), American Mathematical Monthly, vol. 21 (1914), pp. 105–113.

The accelerations of the points of a rigid body (with P. Field), American Mathematical Monthly, vol. 23 (1916), pp. 371–381.

His translations are as follows.

Somoff, *Theoretische Mechanik* (translated from Russian into German). Volume I, *Kinematics*, Teubner, 1879; Volume II, *Statics*, 1879.

P. Duhem, *Émile Mathieu, His Life and Works*, this Bulletin, (1), vol. 1 (1891), pp. 156–168.

F. Klein, *Riemann and his significance for the development of modern mathematics*, this Bulletin, (2), vol. 1 (1894), pp. 165–180.

V. Schlegel, *On the problem of the minimum sum of the distances of a point from given points*, this Bulletin, (2), vol. 1 (1894), pp. 33–52.

F. Klein, *Evanston Colloquium Lectures*, 1894. Published by American Mathematical Society. Reported by A. Ziwet.

He also published numerous book reviews in this Bulletin.

THE UNIVERSITY OF MICHIGAN

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GERMAN EDITION OF DICKSON ON ALGEBRAS

Algebren und ihre Zahlentheorie. By L. E. Dickson. *Mit einem Kapitel über Idealtheorie von A. Speiser.* Zürich und Leipzig, Orell Füssli Verlag, 1927. 308 pp.

Another notable advance in the development of the subject of linear algebras occurred with the appearance of this book. The mathematical world naturally expects a high standard in the case of treatises by Professor Dickson and the present one will not be found disappointing. While to a certain extent a revision and translation into the German language of his earlier book, *Algebras and their Arithmetics* (reviewed in this Bulletin, vol. 30 (1924), pp. 263–270, by Professor Olive C. Hazlett), it contains much material that is new and interesting, particularly along the lines of extension of the theory of division algebras and of the theory of numbers. Included in the latter category is the final chapter on ideal theory by Professor A. Speiser, under whose supervision the translation was made and who has written the preface.

On the basis of the new results contained in this book the first award of the Cole Prize by this Society was made at its meeting in Chicago in April, 1928. It is worth noting also that some years earlier the annual prize of the A. A. A. S. was awarded to Professor Dickson for a paper of unusual merit on the same subject.

It will be sufficient here to discuss the chief ways in which this book differs from the earlier one. Aside from a rearrangement of the material the first thing that strikes the reader's attention is the extension of the theory of linear associative algebras contained in sections 34–44. While based on an article by the author in volume 28 of the Transactions of this Society, it nevertheless differs considerably from that in that it avoids the use of the theory of finite groups. Whether this is really an advantage may perhaps be debatable.

The author deals here with a division algebra A over a general field K that contains a principal unit and that has the four properties: (I) A is of order n^2; (II) it contains an element i that satisfies an equation, $f(\omega)=0$, irreducible in K; (III) the only elements in A that are commutative with i are polynomials in i with coefficients in K; (IV) all the roots of $f(\omega)=0$ are rational functions of i with coefficients in K. In view of a theorem proved later, however, the only condition, as the author points out, that really restricts in any way the generality of the attack on the determination of associative division algebras is the last.

For the case where the irreducible equation $f(\omega)=0$ is "cyclic," to use the terminology of finite groups, algebras of this sort had been determined previously by the author and had been discussed in the English edition. The notable advance here consists in the use of non-cyclic equations. The chief difficulty of the problem lies in the imposition of the associative law, which leads to some conditions that are not particularly simple. Any doubt

that may arise as to the existence of algebras of this type is dispelled however by detailed consideration of special cases that has been carried out. A recent contribution along this line is a Chicago dissertation by A. A. Albert, in which the division algebras of order 16 are determined.

In a later chapter the author discusses generalized quaternion algebras over the field of rational numbers, these being defined by elements i, j that satisfy the conditions $i^2=-1$, $j^2=\tau$, $ji=-ij$, where τ is restricted to certain rational integral values. As in the case $\tau=-1$, which has been treated at considerable length by A. Hurwitz in his book *Vorlesungen über die Zahlentheorie der Quaternionen*, he develops a theory of integral elements of such algebras that parallels closely the corresponding theory for quadratic algebraic fields, allowance being made for the failure of the commutative law. For certain values of τ the analog of the euclidean algorithm for rational integers is shown to hold and a theory of representation of integers by certain quaternary quadratic forms is obtained. The results established here have very likely furnished the inspiration for the recent work by the author and some of his students on the representation of integers by more general forms of this sort. A result of particular interest concerns the solution of diophantine equations of the type $f=y_1y_2\cdots y_k$, where f represents certain quaternary quadratic forms.

The following chapter on the general number theory of algebras has been to a large extent rewritten and supplemented by new and interesting material. This theory, restricted in the English edition to algebras over a rational field, is here extended to algebras over a general algebraic field, being based on independent work in this direction by the author and by Professor Olive C. Hazlett. "Integral elements" of an algebra are here defined as those that satisfy equations with coefficients that are integers in the field over which the algebra is defined and with leading coefficient unity, and that belong to a "maximal integral domain" of the algebra, that is, a set of elements that reproduce themselves under addition, subtraction, and multiplication, and that are not included in a larger domain of the same sort. The second condition is not a consequence of the first, as is the case with algebraic integers, and hence all integral elements of an algebra may not be included in one integral domain. The definition given here seems simpler than that in the English edition where the "rank equation" was employed. As a justification for it there is proved the important theorem that integral elements exist in every algebra.

As in the earlier edition, the theory of integral elements for a general algebra is shown to be associated with that for a semi-simple subalgebra, and the latter to be dependent on the corresponding theory for simple algebras. This in turn leads to the consideration of matrices whose elements are integral quantities of a division algebra. In particular, some interesting theorems are obtained for the case where this division algebra possesses a euclidean algorithm. The theory of the factorization of such matrices leads to a remarkable result concerning the solution, in integers belonging to algebraic fields that possess this property, of an equation of the form $d=y_1y_2\cdots y_k$, where d represents a determinant whose n^2 elements are independent variables. An application is given to the solution of the

equation $x^2+y^2+z^2=u^2+v^2+w^2$ in integers belonging to such a field. It is perhaps worth noting that the values for the six indeterminates obtained by the method might have to be multiplied by 2 (or possibly some factor of 2) in order that they should become integers.

Other new results are a theorem concerning maximal integral domains in a semi-simple algebra, a determination of the bases of all maximal domains of integral quaternions in a quadratic field, a proof that every normal division algebra of order n^2 is of rank n, and a generalization of Cayley's non-associative algebra of order 8. Further additions are a determination of all division algebras of rank 2 and those of order $\leqq 4$, certain theorems concerning algebraic numbers, a discussion of "quasi-fields," including a proof that every finite quasi-field is a field, and theorems concerning the extension of fields by the adjunction of indeterminates and concerning algebras in quasi-fields.

The final chapter on the theory of ideals in rational algebras, which was contributed by A. Speiser, seems interesting and well deserving of inclusion in spite of the fact that the introduction of ideals fails to restore unique factorization in contrast to the situation in algebraic fields. It is necessary here to distinguish between "right-sided," "left-sided," and "two-sided" ideals. The discussion is devoted in considerable part to the set of residues with respect to a two-sided ideal modulus. As these constitute an algebra over a finite field, a number of the fundamental theorems proved in preceding chapters apply to them. In the case of a two-sided prime ideal it is shown that the residues form a system that is isomorphic with respect to addition and multiplication with the totality of all matrices of a particular order with elements in a particular (finite) Galois field. An important theorem on algebraic numbers is generalized by proving that the prime divisors of the discriminant and only these are divisible by the squares of (two-sided) prime ideals.

The book is remarkably free from obscurities or inaccuracies. A few places of this sort in the English edition that the reviewer had noted have in nearly every instance been removed in the revision. An exception is possibly the proof of the theorem (section 71 of the new edition) that, if an algebra contains an idempotent element, it contains at least one such that is primitive. Here it appears to be assumed that the relation, $euAue < eAe$, follows as a consequence from $uAu < A$, a step that does not seem obvious to the reviewer. Only a slight modification of the proof however is necessary to remove this (apparent) obscurity.

In the proof of Theorem 8 in section 81 it may not be evident to a reader what is the relation between $[B]$ and B. In view of the proof on the next page that $A-N=(B-N_1)\times M$, where $N_1=N\wedge B$ is invariant in B, it follows that $[B]=B-N_1$.

The book is very attractive in its make-up, and the translation and proof-reading have been very carefully done. All concerned are to be congratulated on the result.

H. H. MITCHELL

EISENHART'S COLLOQUIUM LECTURES

Non-Riemannian Geometry. By Luther Pfahler Eisenhart. New York, American Mathematical Society, 1927. viii+184 pp.

Generalization having become a familiar process to mathematicians, it is not strange that Riemannian geometry was hardly sixty-four years old before the invention of the non-Riemannian. The former had its origin in Riemann's Habilitationsschrift (1854); the latter, in an article by Weyl (Mathematische Zeitschrift, vol. 2, 1918). Just a decade after the creation of non-Riemannian geometry Professor Eisenhart has given in his colloquium lectures an account of the essential features of the new theory with special reference to the lines along which it has been developed in this country, notably at Princeton.

The fundamental idea of non-Riemannian geometry is the association with a given manifold of a connection which forms a basis for comparing vectors at different points. In the first chapter of Eisenhart's book the asymmetric type of connection is studied, and parallelism of vectors is discussed. The author is careful to distinguish between vectors arising by parallel displacement and parallel vectors: given a vector v at a point x and a curve C passing through x, there is at any point x' of C a unique vector v' which arises by parallel displacement of v along C, whereas all vectors $\phi \cdot v'$, where ϕ is an arbitrary factor and v' is the vector just defined, are called parallel to v with respect to C. The definition of parallel displacement, which is given only for a curve, might well have been stated in the following more general form: the vector v undergoes parallel displacement as x describes any integral variety of the differential equations

$$v^i{}_{,j}\, dx^j = 0,$$

where $v^i{}_{,j}$ denotes the covariant derivative of v. If the vector field is assigned a priori so that $|v^i{}_{,j}| \neq 0$, the integral variety referred to in the suggested modification of the definition is a point, and the vector actually undergoes no displacement. If the connection is euclidean and the coordinates cartesian, the integral variety is the whole of space for any vector field with constant components. From the notion of parallelism are obtained the equations of the paths of the space, a path being defined as a curve whose tangents are parallel with respect to the curve. The paths are thus the straightest lines of the space.

In §8 incompletely integrable systems of total differential equations are discussed. The author proves a theorem which is fundamental in the subsequent treatment (Chapters II and III) of the equivalence problems. The later repetition of this theorem adapted to the particular problem in hand becomes rather monotonous. It would perhaps be sufficient to set up the integrability conditions in each case and to make a concise reference to the original statement of the theorem.

The remainder of Chapter I is devoted to the parallel displacement of

a vector around an infinitesimal circuit, orthogonality of vectors, the generalization of the coefficients of rotation of an orthogonal ennuple, and the changes of connection which preserve parallelism. Of these topics, the last mentioned is perhaps the most important, since it leads to the projective theory of Chapter III.

Most of the results obtained for the asymmetric case to date are such direct generalizations of the symmetric or of the Riemannian case that their acquisition is not a great achievement. It is well, however, to develop the theory as far as possible without making the assumption of symmetry.

Chapter II deals with symmetric connections. Affine normal coordinates are defined in the usual way by means of the equations of the paths, but aside from an application to the definition of normal tensors and the process of extension, no subsequent use is made of them. If one were to judge from the way in which investigators fail to employ normal coordinates as a tool, one would conclude that they are beautiful but useless. In fact, little knowledge has been acquired so far by their use which cannot be as easily obtained with geodesic coordinates. There is an error in formula (22.8). For the correction, see Annals of Mathematics, (2), vol. 28, p. 556.

The theorem of Fermi to the effect that the Christoffel symbols are zero along any given curve in a properly chosen coordinate system is extended to symmetric connections. The proof is unnecessarily long. The following much simpler proof is indicated by Cartan (Mémorial des Sciences Mathématiques, No. 9, p. 18). Without loss of generality we may assume that $x^2=x^3=\cdots=x^n=0$ along the given curve. Assume for the new coordinates x' a transformation which is quadratic in $x^2, x^3, \cdots, x^n$ and which has for coefficients undetermined functions of x^1. Substitute these values of x' together with $x^2=x^3=\cdots=x^n=0$ and $\Gamma'=0$ in equations (28.1) in which the primes have been shifted:

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^k} + \Gamma'^i_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^j}\frac{\partial x'^\beta}{\partial x^k} = \Gamma^\alpha_{jk}\frac{\partial x'^i}{\partial x^\alpha}. \tag{28.1}$$

The coefficients of the zeroth and first orders are then determined by a system of ordinary differential equations to which Cauchy's existence theorem is applicable, and the coefficients of the quadratic terms are determined by finite relations. This same method can be applied to show that the coefficients of projective or conformal connection can be made zero along a curve. It is quite natural to ask, can we go still further and make the Γ's zero over a surface? The answer is in the negative. For taking the surface as $x^3=x^4=\cdots=x^n=0$, we must find a transformation to satisfy, among other conditions, the relations

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^1} = \Gamma^l_{j1}\frac{\partial x'^i}{\partial x^l}, \qquad \frac{\partial^2 x'^i}{\partial x^j \partial x^2} = \Gamma^l_{j2}\frac{\partial x'^i}{\partial x^l},$$

in which $x^3, x^4, \cdots, x^n$ have been put equal to zero. Differentiating the first of these with respect to x^2 and the second with respect to x^1, and comparing, we find that B^i_{j12} must vanish for $x^3=x^4=\cdots=x^n=0$, B representing the curvature tensor. Hence the transformation does not always exist.

The problem of Fermi and related problems, such as that of local coordinate systems, can be regarded as modified forms of the equivalence problem in which the equivalence is not to hold throughout an n-dimensional region, but only in a sub-space defined by certain relations among the independent variables. The equivalence theorem proper, which gives a necessary and sufficient condition that equations (28.1) have a solution $x'=f(x)$ satisfying them identically in all the independent variables, is discussed in detail in §28. The proof given is for symmetric connections, but the extension to the Riemannian and asymmetric cases is clearly indicated.

Chapter III concerns the projective geometry of symmetric connections. In the reviewer's opinion the theory is not presented in its most symmetrical or attractive form because the author does not free himself from the affine connection. The whole subject can be developed as the theory of a projective invariant, the projective connection. When it is so developed, the conditions obtained are in obviously projective form, involving only the Π's and their derivatives. Formulas (34.3), for example, appear on the surface to express affine conditions because they involve the affine connection, but actually they are projective. The condition obtained in the projective fashion would be the expression defined by (35.17) equated to zero. Even equations (34.8), when needed, can be immediately deduced from (35.12) by an appeal to the notion of normal affine connection (p. 105).

The final portion of the chapter is given to the discussion of collineations in affine and projective spaces. By a collineation is meant a transformation which makes the components of the connection exactly the same functions of the new variables as of the old. The analogue for a Riemannian space is a motion or an automorphism of the quadratic form. As the author remarks, the problem of determining whether a given connection admits collineations is a particular case of the equivalence problem, and can be treated as such. The author chooses, however, to use the method of infinitesimal transformations. Except for the flat case and the rather obvious theorem of page 128, the results are not definitive. They are conditional upon the existence of solutions of certain differential equations. The use of Lie's method might, therefore, seem to have no advantages over a consideration of the system of total differential equations which express the law of transformation of the connection. On the contrary, the advantages of the method employed are two: first, the equations of condition are rendered linear; second, variety of method is secured by the introduction of continuous groups. For $n=2$ a classification of all spaces admitting continuous groups of collineations might be possible and interesting.

Chapter IV develops the geometry first of hypersurfaces and then of the general sub-spaces. The generalized equations of Gauss and Codazzi are developed for both cases. The most noteworthy result of this chapter is perhaps that cited by the author in his preface: "For a sub-space of a Riemannian space there is in general an induced metric and consequently

an induced law of parallelism. There is not a unique induced affine connection in a sub-space of an affinely connected space."

The book ends with a bibliography of the books and articles to which reference is made in the text. Eisenhart's previous books have been conspicuous, among other things, for their carefully prepared and useful indexes. The omission of such an index from the present work is to be regretted. With the growth of mathematical literature, it is becoming more and more inconvenient to be forced to leaf through a whole book to locate a desired bit of information.

Those wishing an introduction to the subject will find *Non-Riemannian Geometry* a useful book. A familiarity with the methods of tensor analysis is, of course, presupposed on the part of the reader. The subject matter is carefully presented and the manipulations are easy to follow. The theorems are clearly stated and are proved in a straightforward manner, a number of the proofs being original with the author. The chief defect in the book—and this seems serious in a set of colloquium lectures—is the absence of any attempt to comment upon the significance of the results or to point out possible lines for future development.

J. M. THOMAS

THREE BOOKS ON DIFFERENTIAL EQUATIONS

An Elementary Treatise on Differential Equations and their Applications. New edition, revised and enlarged. By H. T. H. Piaggio. London, G. Bell and Sons, 1928. 18+256+27 pp.

Gewöhnliche Differentialgleichungen. By J. Horn. Zweite, völlig umgearbeitete Auflage. (Göschens Lehrbücherei.) Berlin, Walter de Gruyter, 1927. 8+197 pp.

Ordinary Differential Equations. By E. L. Ince. London and New York, Longmans, Green, 1927. 8+558 pp.

Professor Piaggio's *Differential Equations* was first published in May, 1920, and was reprinted four times during the next six years. "The object of this book is to give an account of the central parts of the subject in as simple a form as possible, suitable for those with no previous knowledge of it, and yet at the same time to point out the different directions in which it may be developed." The only previous knowledge assumed is that of the differential and integral calculus. The style is admirably adapted to a text for beginners and the large number of examples with answers furnishes adequate drill material.

The usual standard forms of ordinary and partial differential equations occupy the greater part of the book: one chapter is devoted to numerical approximations and another to existence theorems. In the revised edition more examples have been included and a new chapter "Miscellaneous Methods" has been added, dealing with a number of disconnected topics, which may be regarded as in the nature of supplementary reading. The author calls particular attention to some difficulties in the theory of singular solutions, for which he is indebted to some unpublished work by Mr. H. B. Mitchell, formerly Professor at Columbia University.

Professor Horn's book is an abbreviation of his earlier work *Gewöhnliche Differentialgleichungen beliebiger Ordnung*, which was published as volume 50 of the Sammlung Schubert and is now out of print. It may be described as an introduction to the theory of ordinary differential equations, although one of the nine chapters is devoted to standard methods of integration and another to numerical and graphical approximations. The author assumes only an elementary knowledge of the theory of functions of a complex variable and has been guided in his choice of subject matter by a desire to lead the student directly to those topics which have most useful applications in physics and astronomy.

Professor Ince's treatise represents a notable addition to mathematical literature in English. The aim of the author has been to "introduce the student into the field of ordinary differential equations and thereafter to guide him to this or that standpoint from which he may see the outlines of unexplored territory." To quote further from the preface: "For some time I have felt the need for a treatise on differential equations whose scope would embrace not merely that body of theory which may now be regarded as classical, but which should cover, in some aspects at least, the main developments which have taken place in the last quarter of a century. During this period no comprehensive treatise on the subject has been published in England, and very little work in this particular field has been carried out; while, on the other hand, both on the Continent and in America investigations of deep interest and fundamental importance have been recorded."

The book is divided sharply into two parts of about equal length. Part I deals with differential equations in the real domain and Part II in the complex domain. Beginning with elementary methods of integration, the author passes on to the existence and nature of solutions and then to an account of continuous transformation-groups. The remainder of Part I is devoted to linear differential equations in real variables. The first three chapters of Part II discuss the existence theorems and singularities of non-linear equations and these are followed by seven chapters on linear equations in complex variables.

To describe in detail the topics discussed would be to give a résumé of the important articles on differential equations which have appeared in the journals during the last twenty-five years. The book forms an excellent account of the progress during that period. Necessarily some parts of it are rather condensed and the average reader will need to have his pencil and paper at hand in order to follow the argument intelligently. However, the references to original articles and standard texts are sufficiently complete to enable one to look up readily further discussion of every major point.

As an exposition of the present state of the theory of ordinary differential equations the book deserves the highest praise.

W. R. LONGLEY

COURNOT ON MATHEMATICAL ECONOMICS

Researches into the Mathematical Principles of the Theory of Wealth. By Augustin Cournot, 1838. Translated by Nathaniel T. Bacon, with an Essay on *Cournot and Mathematical Economics* and a *Bibliography of Mathematical Economics*, by Irving Fisher. New York, The Macmillan Company, 1927. xxiv+213 pp.

Marshall, in his celebrated treatise,* gives a remarkable estimate of two economists of the early part of the nineteenth century. Everywhere he pays tribute to Ricardo's skill in deductive reasoning and his power of abstraction in making hypotheses. "Ricardo himself had no mathematical training. But his instincts were unique; and very few trained mathematicians could tread as safely as he over the most perilous courses of reasoning. Even the acute logical mind of Mill was unequal to the task" (p. 836). And of Cournot he writes: "· · · Cournot's genius must give a new mental activity to everyone who passes through his hands · · · " (p. xi) and "In Cournot it [the French school] has had a constructive thinker of the highest genius" (p. 766). Ricardo's work, to a mathematician, suggests the additional clarity and completeness which could be obtained by means of mathematical formulation; and Cournot, in the book under review, actually carries out a systematic mathematical discussion of certain important questions in economics "rather to present a few new views than to arrange truths already sufficiently known" (p. 164). The reviewer is able to add his testimony that from Ricardo he received the inspiration to a mathematical study of economics, and in this book of Cournot,† which he still regards as the best elementary treatise on the theory of economics, found such a study already well developed.

This is not to say that the book is perfect. In particular, the last chapter, on *Variation in the Social Income, Resulting from the Communication of Markets*, seems not to be adequate in its analysis, for changes in the *form* of production, resulting from such communication, are not carefully considered. And there are occasional slips, usually not of serious character, which in this edition are noted or corrected. Nevertheless the reviewer maintains his opinion that the reader of inquiring mind will get more out of this book than from any other yet published on the subject.

Moreover Cournot is almost alone in holding to a clear realization of the difference between measurable and non-measurable quantities, pointing out in Chapter 1 that many of the controversies in the subject arise from ambiguities of this sort. In the next to the last chapter this clearness is manifest, for the author is able to build up a serviceable theory of the "Social Income," on the basis of money value.

On the other hand, one recent book on the mathematical principles of

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* *Principles of Economics*, London, 1920.

† The first French edition, in the Harvard College Library.

economics, typical of many others, builds its theory on the following basis.*

"Write $U(x, y, \cdots)$ for an algebraic function of measurable quantities $x, y \cdots$; let it be so related to an entity we will call $S(x, y, \cdots)$, where S is not a calculable function but the non-measurable satisfaction derived from quantities $x, y, \cdots$, that the following postulates are satisfied.

"*Postulates.* (1) When $x, y, \cdots$ vary without affecting the value of $U(x, y, \cdots)$, more x balancing less y, etc., $S(x, y, \cdots)$ remains unchanged.

"(2) When $x, y, \cdots$ vary so as to increase $U(x, y, \cdots)$, $S(x, y, \cdots)$ increases, and if U decreases, S decreases.

"(3) When there are successive variations of $x, y, \cdots$, the first increasing U from U_1 to U_2, the second from U_2 to U_3, so that the second increase is greater than the first $(U_3 - U_2 > U_2 - U_1)$, then the second increase in S is greater than the first; the postulate still to be true when less is written for greater."

Apparently this other author is unaware that he is begging the question. If loci of indifference are expressed by Pfaffian differential equations it does not follow that there is any function of which these are the level loci, for such equations are not necessarily completely integrable. The question is not of names, but of existence. These supposedly general treatments are much more special than their authors imagine.†

To carry the point further, we suggest that most nouns that denote values may be regarded as misnomers. They have possibly a significance "im kleinen"—a local significance,—derived from the local integration of the equation, but no significance in extenso. Not only in ethics and aesthetics would an adjective be "true-er" than a noun; it may be presumptuous even to insist, in the words of a recent controversy on a point in the teaching of the calculus, "that there is only one truth." With that remark, and the advice to economists to follow Cournot, rather than Jevons and his school, in this respect, these problems may well be left to the more experienced consideration of philosophers.

Besides the general problems discussed in the first two and last two chapters, to which we have already alluded, Cournot's book deals in considerable detail with the subjects of foreign exchange, law of demand, monopoly, competition, competition among a large number of producers, producers who furnish parts of a compound product, and the communication of markets. The effects of taxation are investigated with respect to the various forms of production.

Throughout, the problems are analyzed as far as possible in terms of functions of a single variable. Generalizations in this respect are important, but not so difficult for a mathematician to make as for the reader to follow, if he is of the type for which this book was intended. Such generalizations, among other contributions, have been made by Walras and Pareto. Still more significant extensions would arise from the abandonment of the hypothesis, which Cournot and almost all other economists make

* Bowley, *Mathematical Groundwork of Economics*, Oxford, 1924, p. 1.

† The trouble usually comes from reasoning about n dimensions with the intuition merely of a two-dimensional figure.

either explicitly or tacitly, that demand is a function of price, adequately represented by the notation $D=f(p)$. Cournot is also not alone in considering profit as the central quantity in economics, whose maximum determines the other quantities; and given the industrial habits of his time, he is hardly to be blamed.

The translation seems to be carefully done, and the editor has added numerous notes designed to assist the less mathematical reader. No attempt has been made to bring the bibliography down to the date of this edition; in the thirty years since the treatise first appeared in English the additions to the bibliography included in that edition would presumably have surpassed the desirable limits for the book. There would have been service rendered, however, if the editor had used the resources at his disposal to clear up the ambiguous character of some of the references already given. This is not said to diminish the gratitude which the public owes to both editor and translator in making again accessible, after ninety years, the eternal freshness of this brief classic.

G. C. EVANS

THREE BOOKS ON NON-EUCLIDEAN GEOMETRY

Vorlesungen über Nicht-Euklidische Geometrie. By Felix Klein. Prepared anew for publication by W. Rosemann. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 26. Berlin, Julius Springer, 1928. xii+326 pp. Price 18 marks unbound, 19.50 marks in linen.

Nicht-Euklidische Geometrie. Hyperbolische Geometrie der Ebene. By Richard Baldus. Sammlung Göschen, No. 970. Berlin and Leipzig, Walter de Gruyter, 1927. 152 pp. Price 1.50 marks.

La Géométrie Non-Euclidienne. By P. Barbarin: Third edition, followed by notes by A. Buhl. Collection "Scientia," No. 15. Paris, Gauthier-Villars, 1928. 176 pp.

In a note to his Erlanger Programm, Klein justified his avoidance of the word "non-euclidean" in that address, saying: "There are connected with the designation 'non-euclidean geometry' a number of unmathematical conceptions which occasion as much enthusiasm on the one side as horror on the other, but which have nothing whatever to do with our purely mathematical considerations." This may be a reason why, aside from a few glances forward, that same word is absent from over one half of his admirable treatise, *Vorlesungen über Nicht-Euklidische Geometrie.* In fact, the reader with a purely euclidean past will be led so tactfully and persuasively from his usual geometry to the more general one that he may long wonder wherein lies the book's right to its title. He will be an admirer of the greater beauty and symmetry of non-euclidean space without realizing how far he has left euclidean space behind him.

Part one of the book is an introduction to projective geometry. We pass rapidly from elementary affine coordinates in euclidean space to homogeneous and general projective coordinates. Thus, the arithmetical basis

of cartesian geometry being assumed, no time need be given to axioms. Analysis situs enters early, when the projective plane is shown to be one-sided. The chapter on fundamental concepts treats, further, of homogeneous linear substitutions and projective transformations (the Erlanger Programm is to be our guide), n-dimensional manifolds, duality, cross ratios, imaginary elements. There follows a careful treatment of configurations of second degree (both class and order), a complete classification from the point of view of reality being given. A number of transitions from one non-degenerate form to another through a degenerate one are traced—such, for instance, as will later clarify the position of euclidean geometry between its two rivals. The last chapter on projective geometry deals with those collineations which leave invariant a configuration (degenerate or otherwise) of second degree. The fate of points, the families of invariant surfaces, the interpretation of real collineations as motions,—rotations, translations, screw-motions—are studied clearly and fully, for various types of fundamental configurations.

The second part deals with measurement, on a projective basis. Again taking thought that the reader begin with the familiar, Klein discusses the euclidean formulas for lengths and angles, and shows their projective relation to the circular points (or fundamental circle, if there are three dimensions). Then the metric of the bundle of lines or planes (e.g., in euclidean space) gives the first glimpse of measurement in elliptic and spherical geometries. It is noted, as a matter of interest, that elliptic geometry prevails on the euclidean plane at infinity. Only in the fifth chapter is a projective coordinate system set up in its own right, by the familiar repetition of the harmonic construction. A notable point in Klein's treatment is his insistence that coordinates can be set up by constructions entirely within a finite region. The way is then clear for metrical definitions on the basis of the absolute (first non-degenerate, then degenerate) by means of the logarithm of a cross ratio. Now a selection has to be made, among all the possible fundamental configurations, in order that angular measurement be elliptic; and there result, of course, the three ordinary types. The development of their distinctive properties is full and clear. Among the sections of unusual interest may be mentioned the equation, valid in all three geometries, between the dihedral and the trihedral angles of a tetrahedron and the analogous equation in n dimensions; the use of quaternions to represent motions in elliptic space; the excellent study of the Clifford surface and its euclidean geometry. Chapter IX, the last one of this part, discusses those forms of space which admit of elliptic, hyperbolic, euclidean metric. This discussion of analysis situs will probably be among the most influential sections in clarifying geometrical ideas. In the last paragraph we read: "From the assumption that the space about us displays a euclidean or a hyperbolic structure, it by no means follows that this space has infinite extent."

Chapter X is concerned with history, with differential geometry, and with a variety of conformal representations of non-euclidean planes on the euclidean. Finally, a few pages are given to applications of non-euclidean geometry—the use of hyperbolic motions in the study of linear

substitutions of a complex variable and in that of automorphic functions, the use of non-euclidean geometry in topological study, and of projective metric in special relativity. These chapters are brief—probably no fault, for the book has pictured the subject so alluringly that the references here given should lead many students on to wider reading than one treatise could furnish.

There is scarcely an improvement which the reviewer could suggest, even in such a small matter as proof-reading. One sentence, on page 26, with regard to those collineations of a line which result from two real projections, seemed at first reading self-contradictory: "Die so erhaltenen Kollineationen besitzen stets einen reellen Fixpunkt, der aber · · · im allgemeinen nicht vorhanden zu sein braucht." Clarity results if we change to "in den allgemeinen Kollineationen." On page 116, line 9, we should read "nullteiligen und ringartigen Flächen," not "nullteiligen und ovalen Flächen"—the only error of statement noted.

It is surprising that the two words "complex" and "imaginary" are used indiscriminately, whether real numbers are included in the domain intended or excluded from it. A few random cases are the following:

Page 73, line 4, "imaginary" collineations may be real.

Page 93, line 21, "complex" collineations may be real.

Page 93, line 24, "imaginary" point-pairs may not be real.

Page 100, footnote, "complex" values may not be real.

Page 185, line 15, "imaginary" region includes real elements.

Page 236, line 13, "imaginary" region includes real elements.

Page 236, line 18, "imaginary" lines may not be real.

Are not imaginary numbers those complex numbers which are not real?

A similar oversight occurs when Clifford parallels are discussed. On page 234 we read "A Clifford parallel to a line a is never in a plane with a, but always skew to it." In order that this be true, it is necessary that parallels be real. Yet, on page 237, imaginary lines, generators of the fundamental null surface, are made members of a family of parallels, and we have therefore intersecting Clifford parallels. Perhaps it would be best not to name each of the two displacements whose product is a general motion a "Schiebung längs einer *Parallelenschar*."

It may be due to a desire for brevity that, although curvature is carefully explained (p. 280), torsion is soon afterwards mentioned without definition (p. 305), and that a number of important theorems are not proved —for instance, that of the uniqueness of the common self-conjugate triangle of two conics. These are very slight departures from a fulfillment of the evident desire to write a book comprehensible immediately after a first course in calculus.

The value of the treatise, great in any case because of its elegance and comprehensiveness, is enhanced by a profusion of historical notes, by a generous use of cross references, looking both forward and backward, and by such a large collection of clear illustrations as no other work on non-euclidean geometry approaches. A few suggestions for future research are given. Thus: "a closed three-dimensional hyperbolic space-form of finite volume does not seem to have been found as yet"; "surfaces [of constant

curvature] are fixed by difficult differential equations, which have as yet been solved only in particularly simple special cases."

The work has been prepared for the press by Rosemann. As the editor states in the preface, he gave it years of unselfish labor, first in consultation with Klein, then alone. That there resulted such a valuable contribution to mathematical literature is cause for deep gratitude to him.

Although the book of Baldus on non-euclidean geometry, in the Sammlung Göschen, is much smaller, he has, by restricting himself to hyperbolic plane geometry, been able to give an excellent presentation.

The theory common to euclidean and hyperbolic geometries is developed as far as possible, largely on the same basis as in Hilbert's *Grundlagen der Geometrie*. Often, indeed, the reader is referred to that book for proofs of familiar theorems. In this development, elliptic geometry is excluded by the axiom: "Among three points of a line one and only one lies between the other two." Next it is shown that the axioms preceding that of parallelism are valid if we use the unit circle as the fundamental conic, and interpret congruence by means of automorphic collineations of that circle and the preservation of the usual cross ratio. There follows the hyperbolic axiom, in the following frugal form: "There exist one line g and one point P outside of g, such that (at least) two lines through P fail to meet the line g." Angles obtain measure, in this interpretation, without appeal to the cross ratio made with imaginary lines. For this purpose, an angle whose vertex is at the center is given its euclidean measure, while the hyperbolic magnitude of another is defined as that resulting after a "motion" has brought the vertex to the center. Indeed, the device of placing an important point of a configuration at this center, and then applying euclidean theorems in order to obtain hyperbolic ones, is used ingeniously many times. When length has been defined as half the logarithm of the cross ratio, the material is on hand for a good treatment of trigonometry and elementary analytic geometry, of properties of lines and the various conics of area. It is interesting to note that, although the concurrence of the medians of a triangle holds in hyperbolic geometry, the proof is via euclidean geometry and that (according to Baldus) no proof independent of the parallel axiom has been discovered.

Baldus, like Klein, has a satisfactory historical account. (In fact, the reviewer recalls no other branch of mathematics in which the reader is so sure to be entertained by the wisdom and errors of the past.) Much of the booklet's value lies in the philosophic quality of the author's thought. Particularly good are the sections on the significance of "absolute" geometry and on the formalization of geometry, and his reference to the doubts which might arise from the use of a euclidean circle as the basis for hyperbolic geometry, as well as the answer to those doubts.

Figures are abundant, very clear and carefully drawn. There is a short reference list—eleven books—at the beginning. All the works named are available in German. Perhaps, if the author had realized how welcome are the worthy, inexpensive members of the Sammlung Göschen in non-German parts of the world, he would have included such important treatises as those of Coolidge and MacLeod.

The third work on non-euclidean geometry which has come to hand in these days is the third edition of Barbarin's *La Géométrie Non-Euclidienne*. The first edition, in 1902, was a slight book, following the historical line of development. Later editions, in 1907 and 1928, have been rewritten and expanded as other related disciplines developed and as Barbarin's own researches were made and published in scattered journals. Now the volume, more than double its original size, contains, also, important supplementary notes by Buhl. There results a book written, in turn, for three publics: for those who, with scarcely higher mathematics than trigonometry, would learn the essential elementary facts of these geometries; for those who are interested in the more difficult study, though with simple tools, of questions of construction and the like; and for those who are anxious to learn, from a very competent writer, of the relations between non-euclidean geometry and some recent work in physics, in analysis, and in differential geometry.

The historical pages of this book are made more interesting by a number of portraits, more valuable by the inclusion of several theorems and proofs. Since the elementary point of view, that of the searchers after a proof of the parallel postulate, is maintained, there cannot be the same elegant symmetry as projective geometry grants. On the other hand, spherical geometry (which Barbarin prefers to elliptic) and hyperbolic geometry are presented to us somewhat symmetrically. Of two postulates—namely, the sixth (that two lines cannot enclose a space) and the fifth (in Euclid's form)—each is fulfilled by one and only one of the two theories. It is worth while to emphasize that Euclid's parallel postulate is true in Riemann's geometry. Unusual emphasis is placed on de Tilly, particularly on his proof that we must have one of three identities between the coordinates of five points in space,—identities which involve the distances of all pairs of points, and lead respectively to the three geometries. Those of us who prefer the projective approach to the subject will perhaps not rank de Tilly's work as highly as Barbarin does; nevertheless it is well that stress is laid on his researches. Another seemingly excessive enthusiasm is that for Euclid's *Elements*: "Leur emploi universel des arguments les plus complets et les plus rigoureux;" "les postulats d'Euclide sont absolument rigoureux"—it is hard to say what is meant by rigor of axioms.

The other chapters which remain from the first edition summarize the main facts on lines and the various types of circles, develop trigonometric formulas of triangles and trirectangular quadrilaterals, discuss areas and volumes. It is curious that the author does not notice that, in Riemannian geometry, equidistant curves (his "hypercycles") are circles. Unfortunately, proofs are so often absent that the reader may not be convinced.

Barbarin points out that Euclid's reasoning is always by means of figures of which the construction is known. He has followed this clew, and has interested himself in the discovery of methods of construction in non-euclidean geometry. Thus, the new chapters describe a mechanism for tracing horocycles; the construction of angles of rational trigonometric functions, of line segments whose trigonometric (or hyperbolic) functions

are rational, or roots of index 2^n, of rational numbers; inscription of regular polygons; the quadrature of the circle.

A chapter on the impossibility of proving the fifth postulate takes time to refute some very crude objections to the geometries of Lobachevsky and Riemann, discusses the surfaces on which their plane geometries are valid,—and, incidentally, contains a welcome photograph of Beltrami's paper pseudosphere.

With the supplementary notes of Buhl, who wrote the pamphlet on "Formules Stokiennes" of the Mémorial, we seem to begin a new book, one on those formulas. (Neither here nor in the Mémorial does he trouble to define the adjective.) The second note, starting with the derivation of Maxwell's equations from a Stokesian one, passes soon to the problem of the linear transformations leaving a homogeneous quadratic function invariant—that is, to that Cayleyan problem which was at the basis of Klein's treatise. On this basis, then, Buhl develops non-euclidean geometry anew. Twenty pages are next devoted to the much more general differential geometry of Riemann, once more with a start from Stokes. For its brevity, this is a remarkably good account; but it would be clearer if the author did not fear the words covariant and contravariant, and avoid all explanation of the distinction between subscript and superscript. The fourth note, entitled *The Geometry of Light*, deals condescendingly with restricted relativity.

Probably the simultaneous publication of several works on non-euclidean geometry is due, in part, to the influence of recent physical theories. The three books here reviewed are well fitted to lead to keen interest in that subject for its own sake.

E. S. ALLEN

SHORTER NOTICES

Précis d'Analyse Mathématique à l'usage des candidats au certificat de calcul différentiel et intégral. By E. Lainé. Vol. I, viii+231 pp.; Vol. II (in collaboration with G. Bouligand), 315 pp. Paris, Librairie Vuibert, 1927.

Among the large number of the French textbooks with the similar purposes, this is perhaps one of the most interesting. A definite and, in our opinion, a fortunate break with the old tradition is made by introducing the vector notation in the treatment of the fundamentals of differential geometry. The corresponding part (iv) of the book as well as the part (v) devoted to the partial differential equations (and written by G. Bouligand) are the best in the book. The usage of the vector notation makes intuitive and simple many results which otherwise are obtainable but with a great labor. The inductive method is also used in the treatment of the partial differential equations (the only ones being treated are those of the first order and some of the second order, of the Monge-Ampère type). The reader is compelled here to make his own conclusions on the basis of intuitive geometric discussion.

As to the remaining parts (Introduction: Review of some facts of algebra and analytic geometry; Part i: Functions of a real variable; Part ii: Analytic functions; Part iii: Ordinary differential equations), they reveal the tendency of putting too much material into too limited space (279 pp.). The resulting treatment is often rather brief and formal, and sometimes a little confusing. As for examples we may mention: infinite double integrals (Vol. I, 69–70); line and surface integrals (I, 76–77); extrema of functions of two variables (I, 82–83); differentiation under integral sign (I, 107). There are found some lapses and quite a few misprints. The text is supplemented by a large number of exercises; some of them, interesting as they are, appear rather difficult from the point of view of the means available in the text.

J. D. TAMARKIN

The Early Mathematical Sciences in North and South America. By Florian Cajori. Boston, Richard G. Badger, 1928. 156 pp.

Excluding introductory matter and a six-page index, this book consists of eleven chapters distributed over 139 pages as follows: Mathematics, 18 pp.; Practical Astronomy and Surveying, 37 pp.; Surveys and Maps, 24 pp.; Meridian Measurements of the Earth, 8 pp.; Transit of Venus, 10 pp.; Comets, 5 pp.; Almanacs, 5 pp.; Orreries, 3 pp.; The Earliest Permanent Observatory in America, 8 pp.; Physics, 16 pp.; Societies, Academies and Journals, Conclusion, 5 pp. As may be inferred from these chapter headings, the work appears to be a series of notes or sketches on topics more or less closely related to mathematics, physics and astronomy rather than a systematic study of the development of these sciences

as sciences in North and South America during the sixteenth, seventeenth and eighteenth centuries. This inference becomes more apparent upon noticing the sub-headings in Chapter I, Mathematics: The Maya symbol for zero; The Peruvian and North American knot records; The calderón and cifrão in the writing of numbers; evolution of the dollar mark; magic squares; mathematical publications in America.

The fact that such paragraph titles as "Brattle using Halley's quadrant at Harvard" (p. 42), "Brattle and Kearsley using ring-dials" (p. 44) and "Laval at the gulf coast" (p. 43) occur along with "Early determinations of positions in and near Mexico" (p. 36) and "Marcgrave's observations in Brazil" (p. 41) suggests that the author has not given the form of this work the careful revision his other publications would lead us to expect.

Typographical errors are not unusually numerous but those which the reviewer has noticed lead him to wonder if the proof-reading has not been done chiefly in the publisher's office. For example, on p. 12 "Axtecs" appears in place of "Aztecs"; a reference to Hevelius on p. 46 is referred to in the index (p. 153) as "Hevelins"; under B in the index (p. 151) occurs "Brahe. See Tycho Brahe" and under T (p. 156) is found "Tycho, Brache" instead of "Tycho Brahe"; and although Professor J. L. Coolidge of Harvard must be well known personally to the author his name is cited as "J. J. Coolidge" in three different places (pp. 26, 28 and 152).

The mechanical makeup of the book is scarcely what might be expected from the Gorham Press of Boston. On various pages (e.g., pp. 74, 77, 93, 111, 124) certain lines stand out prominently and, in some places, the contrast is almost sufficient to give the appearance of a line of boldface type. Whether this variation comes from setting the text on one linotype machine and the corrections on another or from other causes, it is so easily corrected by a competent pressman that its appearance suggests careless or hurried presswork. Also, the reviewer has an impression that the illustrations which appear on plate paper might have been improved, both in appearance and in placing on the pages, without adding unduly to the cost of publication.

The reviewer has found the book quite interesting and, so far as he is able to judge, its statements of fact have been carefully verified.

U. G. MITCHELL

NOTES

The American Mathematical Society is a beneficiary under the will of the late Miss Marion Reilly, of Bryn Mawr College, as residuary legatee of a trust fund.

The opening number of volume 31 of the Transactions of this Society (January, 1929) contains the following papers: *The boundary problem associated with a differential equation in which the coefficient of the parameter changes sign*, by R. E. Langer; *Triple and multiple systems, their geometric configurations and groups*, by A. Emch; *On the expansion of analytic functions of the complex variable in generalized Taylor's series*, by D. V. Widder; *Note on the expansion of analytic functions in series of polynomials and in series of other analytic functions*, by J. L. Walsh; *A general problem of minimizing an integral with discontinuous integrand*, by C. F. Roos; *An introduction to the theory of ideals in linear associative algebras*, by C. C. MacDuffee; *On the Padé approximants associated with the continued fraction and series of Stieltjes*, by H. S. Wall; *On curvilinear congruences*, by C. E. Weatherburn; *The multinomial solid and the chi test*, by B. H. Camp; *The singular points of analytic space-curves*, by A. Ranum; *Universal quadratic forms*, by L. E. Dickson; *Boundary value problems for potentials of a single layer*, by E. R. C. Miles; *On a generalization of the associative law*, by A. Suschkewitsch.

The opening number of volume 51 of the American Journal of Mathematics contains: *Discontinuous boundary value problems of the first kind for Poisson's equation*, by G. C. Evans; *On irreducible cross-cuts of plane simply connected regions*, by W. A. Wilson; *Number of abelian subgroups in every prime power group*, by G. A. Miller; *Finite groups in which conjugate operations are commutative*, by C. Hopkins; *Covariant conditions for multiple roots of a binary form*, by L. T. Moore and J. I. Tracey; *On certain finitely solvable equations between arithmetical functions*, by E. T. Bell; *On the number of representations of integers by certain ternary quadratic forms*, by J. V. Uspensky; *Representation of integers in the form* $x^2+2y^2+3z^2+6w^2$, by L. W. Griffiths; *Normal ternary continued fraction expansions for cubic irrationalities*, by P. H. Daus; *Triads of plane curves*, by A. R. Jerbert; *The symbolic development of the disturbing function*, by K. P. Williams; *An extension of Maschke's symbolism*, by N. L. Anderson; *On the approximate solution of Fredholm's homogeneous integral equation*, by R. B. Adams; *Functions of écart fini*, by H. E. Bray.

The November, 1928, number of the Annals of Mathematics (series 2, volume 29, No. 4) contains: *Concerning the complementary domains of continua*, by G. T. Whyburn; *Fermat's Note* XLV, by C. M. Walsh; *A new derivation of the Lorentz transformation*, by P. Y. Chou; *On m-dimensional cross-ratios*, by L. Weisner; *Twisted curves classified as to their osculating and quasi-osculating spheres*, by A. Ranum; *On the generalization of the algebra of the theory of numbers. The invariancy of infinite series*, by O. E.

Glenn; *Involutions that belong to a linear class*, by E. T. Browne; *Euclidean invariants of plane algebraic curves*, by L. Weisner; *Non-abelian groups whose groups of isomorphisms are abelian*, by C. Hopkins; *Oscillation theorems for the differential boundary value problems of the fourth order*, by S. A. Janczewsky; *A note on Stieltjes integrals*, by R. N. Haskell; *On motions which satisfy Kepler's first and second laws*, by L. R. Ford.

Professor G. H. Hardy, of Oxford University, who has been lecturing at Princeton University during the first semester of the present academic year on *Chapters in the theory of functions*, lectured at Lehigh University on January 11 on *Hilbert's Mathematical logic*, at the Ohio State University on January 18, 1929, on *The theory of primes*, and at the University of Chicago on January, 21 on *The analytic theory of numbers*. From January 27 to March 15, he was in residence at the California Institute of Technology, giving lectures on *The analytic theory of numbers*. On March 1 and 5 he lectured at the University of California at Los Angeles. Between March 7 and March 12 he delivered the Hitchcock lectures at the University of California at Berkeley. As announced elsewhere, he will speak at the invitation of the program committee at the meeting of this Society in New York City on March 29 on *Modern work in the theory of ordinary trigonometric series*.

Professor Hermann Weyl, of the Zurich Technical School, who has acted as research professor of mathematical physics at Princeton University during the academic year, has been appointed as Visiting Lecturer by the Council of the American Mathematical Society for the year 1928–29. (See this Bulletin, vol. 34, p. 22.) Professor Weyl has already delivered lectures in the Universities of Iowa and Michigan (in December), and at Columbia University (in March); and he will make an extended tour during the month of May, travelling slowly from Princeton westward to California, where he will teach in the Summer Session at the University of California at Berkeley. A detailed schedule of all of his lectures will appear later. As announced elsewhere, he will speak at the invitation of the program committee at the meeting of this Society in New York City on March 30, on *Fournier series and almost periodic functions from the standpoint of the theory of groups*.

Professor Arnold Emch, of the University of Illinois, who has been on leave of absence in Europe during the present academic year, lectured before the Mathematical Colloquium of the University of Zürich on January 28 on the topic *Cremona transformations and algebraic curves and surfaces*.

Professor James Pierpont's address on *Mathematical rigor, past and present* (this Bulletin, vol. 34, pp. 23–53) has been reprinted in the Revista Matemática Hispano-Americana, (2), vol. 3 (1928), Nos. 7–8, under the title *El rigor matemático. Pasado y presente*. The translation is by Alvarez Aguirre.

At the annual meeting of the London Mathematical Society, the following officers were elected: president, Professor E. T. Whittaker; vice-presidents, Mr. R. H. Fowler, Professor E. H. Neville, and Mr. E. C.

Titchmarsh; treasurer, Dr. A. E. Western; librarian, Professor Harold Hilton; secretaries, Professor G. N. Watson and Mr. F. P. White; new members of the Council, Professor Oswald Veblen and Mr. T. L. Wren.

Among the officers elected at the New York meeting of the American Association for the Advancement of Science are the following: president, Professor R. A. Millikan; vice-president of Section A (mathematics), Professor E. T. Bell; Secretary of Section A, Professor C. N. Moore; vice-president of Section K (economic and social sciences), Professor H. L. Rietz; secretary of Section K, Professor C. F. Roos.

Dr. Max Mason, director of the division of natural science of the Rockefeller Institute, and Professor M. I. Pupin, of Columbia University, have been appointed members of the National Research Council's committee to cooperate with the officials of the Chicago World's Fair Centennial Celebration. Mr. F. B. Jewett, vice-president of the American Telephone and Telegraph Company, is chairman of this committee.

Dr. Irving Langmuir, of the General Electric Company, has been elected president of the American Chemical Society.

The Paris Academy of Sciences announces the award of the following prizes for 1928: the Poncelet prize to Gaston Julia, for his mathematical work; the Francœur prize to Szolem Mandelbrojt, for his work in mathematical analysis; the Montyon prize in mechanics to Filippo Burzio, of Turin, for his work in ballistics; the Henri de Parville prize to F. C. Haus, of the Belgian Service Technique de l'Aéronautique, for his *Etude dynamique sur la vrille*; the Valz prize to Georges van Biesbroeck, of the University of Chicago, for his astronomical work; the Janssen medal to William Wright, of the Lick Observatory, for his astronomical work; the Plumey prize to Albert Thuloup, for his memoir entitled *Essai sur la fatigue des tuyaux mince à fibre moyenne plane ou gauche*; the Montyon prize in statistics to Georges Darmois, for his work entitled *Statistique Mathématique*; the Grand Prix for mathematical sciences to Georges Giraud, for his work in partial differential equations; the Vaillant prize to Maurice Fréchet, for his work in abstract sets; the Roux prize to François Divisia, for his work entitled *Economique Rationnelle*; a prize from the Gegner foundation to Maurice Vèzes, for his *Traité de Chimie Physique*; a prize from the Hirn foundation to Maurice Gevrey, for his work in partial differential equations; a prize from the Henri Becquerel foundation to Paul Lévy, for his work in the analysis of functionals.

The Royal Society of London has awarded the following medals: a royal medal to Professor A. S. Eddington, for his contributions to astrophysics; the Sylvester medal to Professor W. H. Young, for his contributions to the theory of functions of a real variable; the Rumford medal to Professor Friedrich Paschen, for his contributions to the knowledge of spectra; the Hughes medal to Maurice de Broglie, for his work on X-ray spectra.

The Buchan prize of the Royal Meteorological Society has been awarded to Dr. Harold Jeffreys, for papers contributed to the Quarterly Journal of

that Society during the years 1924–1927, on cyclones, fluid motions produced by differences of temperature and humidity, dynamics of winds, and related subjects.

The American Mathematical Society has awarded its Bôcher Memorial Prize to Professor J. W. Alexander, of Princeton University, for his memoir entitled *Combinatorial analysis situs*, in volume 28 of the Transactions of this Society. (See the present issue of this Bulletin, p. 162.)

The National Academy of Sciences has awarded its Comstock prize, for the most important research in electricity, magnetism, and radiant energy during the past five years, to C. J. Davisson, of the Bell Telephone Laboratories.

Professor A. H. Compton, of the University of Chicago, has been awarded the gold medal of the Radiological Society of America for his work in X-rays.

The New York Academy of Sciences has awarded its A. Cressy Morrison prize to B. P. Gerasimovic, of the Harvard College Observatory, and D. H. Menzel, of the Lick Observatory, for a thesis on *Subatomic energy and stellar radiation*.

Professor Serge Bernstein has been elected correspondent of the Paris Academy of Sciences, in the section of geometry.

Professor G. D. Birkhoff has been elected correspondent of the Paris Academy of Sciences.

Professor G. Koenigs, of the Sorbonne, has been elected an associate of the Royal Academy of Belgium.

Professor A. Fraenkel, of the University of Kiel, has been appointed head of the mathematical institute of the University of Jerusalem.

Associate Professor Georg Joos has been promoted to a full professorship of theoretical physics at the University of Jena.

Professor Adolf Smekal, of the University of Vienna, has been appointed professor of theoretical physics at the University of Halle.

Dr. F. A. Willers has been appointed professor of mathematics at the School of Mines of Freiberg i. Sa.

The following have been admitted as private docents: Dr. A. Berger, for insurance mathematics, at the University of Vienna; Dr. G. Thomsen, for mathematics, at the University of Hamburg.

Mr. Arthur Berry and Mr. H. W. Richmond have retired from university lecturerships at King's College, Cambridge.

Mr. A. S. Besicovitch has been appointed Cayley lecturer in mathmatics at Cambridge University.

Mr. H. S. Ruse has been appointed lecturer in mathematics at the University of Edinburgh.

Assistant Professor Herman Zanstra, of the University of Washington, has been appointed assistant professor of mathematics at the Imperial College, London.

The Institute of Mathematics and Applied Mathematics of the University of Paris, dedicated to the memory of Henri Poincaré, was formally opened on November 1, 1928; Premier Poincaré presided at the ceremonies. Funds for the erection of the building were provided by the Baron Edmond de Rothschild and the General Education Board, the latter subscribing also an endowment for a chair of applied mathematics. See pages 198–200 of this issue.

A memorial building in honor of the late Dean H. B. Fine, to be known as the Henry Burchard Fine Mathematical Hall, will be erected at Princeton University for the use of the departments of mathematics and mathematical physics. It is the gift of T. D. Jones, of the Princeton class of 1876, and his niece, Miss Gwenthalyn Jones.

The Carnegie Foundation has made a grant of $25,000 to Princeton University for the study of light conditions under which atoms exist.

Dr. P. A. M. Dirac, of St. John's College, Cambridge, will lecture on quantum mechanics at the University of Wisconsin during April and May.

Assistant Professor R. W. Babcock, of the University of Wisconsin, has been appointed as professor and head of the department at De Pauw University, to take effect in September, 1929.

Mr. W. M. Bond has been appointed professor of mathematics at Waynesburg College, Waynesburg, Pa.

Mr. Paul K. Rees, of Texas Technological College, has been appointed assistant professor of mathematics at the University of Mississippi; Mr. Thomas A. Bickerstaff has been appointed to a teaching fellowship at the College.

Professor A. A. Titsworth has retired after forty-two years of service in the mathematical department of Rutgers University.

Adjunct Professor G. T. Whyburn, of the University of Texas, has been appointed assistant professor of mathematics at the Johns Hopkins University, to take effect in September, 1929.

Professor G. H. Bryan, retired, of the University of North Wales, Bangor, died October 13, 1928, at the age of sixty-four.

Dr. J. W. L. Glaisher, F. R. S., senior fellow of Trinity College, Cambridge, died December 7, 1928, at the age of eighty.

Professor T. C. Chamberlin, of the department of geology of the University of Chicago, known for his planetesimal hypothesis, died November 15, 1928, at the age of eighty-five.

The Reverend J. N. Donahue, president of Columbia University, Portland, Ore., died recently.

Dr. Leonard Waldo, astronomer and consulting engineer, died January 25, 1929, at the age of seventy-five.

NEW PUBLICATIONS

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FORD (W. B.). A first course in the differential and integral calculus. New York, Holt, 1928. 6+372 pp.

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HILL (T. H. W.). Elementary analytic geometry. London, Mills and Boon, 1928.

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KAYE (G. R.). The Bakhshāli manuscript. A study in mediaeval mathematics. (Archaeological Survey of India, new series, vol. 43, parts 1–2.) Calcutta, Government of India Central Publication Branch, 1927. 4+156 pp.+48 plates.

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MAENNCHEN (P.). Methodik des mathematischen Unterrichts. Frankfurt a. M., Verlag Diesterweg, 1928. 10+297 pp.

MILLOT (S.). La balance à calcul. Paris, Gauthier-Villars, 1928. 20 pp.

MISER (W. L.). See PALMER (C. I.).

MOREAU (G.). See RUSSELL (B.).

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NEUMANN (F.). Franz Neumanns gesammelte Werke. Herausgegeben von seinen Schülern. Band 1. Leipzig, Teubner, 1928. 10+428 pp.

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PETERS (J.). Sechsstellige Tafel der trigonometrischen Funktionen. Berlin, Dümmlers Verlag, 1929. 293 pp.

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Many others are in press or in preparation.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

BERKELEY, CALIFORNIA, June 20-21, 1929.

The previously announced meeting at Stanford on April 6 has been postponed and combined with this meeting, which is to be held in conjunction with the Pacific Division of the American Association for the Advancement of Science, which meets in Berkeley on June 19-22. Abstracts must be in the hands of Professor B. A. Bernstein, University of California, Berkeley, Calif., *not later than May 1*, since it is desired to insert the program in the general program of the A.A.A.S. meeting. Professor James Pierpont will deliver an address at the invitation of the program committee to deliver special addresses.

BOULDER, COLORADO, SUMMER MEETING AND COLLOQUIUM, August 27-30, 1929.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than July 30. Professor Virgil Snyder will deliver his retiring presidential address, entitled *The problem of the cubic variety in four-way space;* and Professor R. L. Moore will deliver a series of colloquium lectures on *Point-set theory.*

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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Changes of address of members should be communicated at once to the Secretary of the Society, R. G. D. RICHARDSON, 501 West 116th Street, New York.

The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. FITE, 501 West 116th Street, New York.

WHOLE NUMBER 359

CONTENTS

PAGE

The Thanksgiving Meeting of the Southwestern Section. By E. B. STOUFFER 145
The Thanksgiving Meeting in Cincinnati. By M. H. INGRAHAM 149
The Annual Meeting in New York. By R. G. D. RICHARDSON 156
The Christmas Meeting in Chicago. By M. H. INGRAHAM 191
A Correction. By W. J. TRJITZINSKY 197
The Inauguration of the Institute Henri Poincaré in Paris. By MAURICE FRÉCHET 198
Report on the 1928 International Congress of Mathematicians. By L. TONELLI 201
Maxime Bôcher. By W. F. OSGOOD 205
On Regular Points of Continua and Regular Curves of at Most Order n. By G. T. WHYBURN 218
Cubics Whose (Hessian) ^{2}'s Are Themselves. By F. E. WOOD 225
A Proof of the Fundamental Theorem of Algebra. By I. M. SHEFFER 227
An Application of Tensor Analysis to the First Variation of an Integral. By J. H. TAYLOR 231
On the Maximum Defficiency of r-Space Curves. By B. C. WONG 237
A Note on Certain Continuous Non-Differentiable Functions. By F. W. PERKINS 239
A New Proof of Certain Relations of Morse in the Calculus of Variations in the Large. By D. E. RICHMOND 248
Alexander Ziwet—In Memoriam 259
German Edition of Dickson on Algebras. By H. H. MITCHELL 261
Eisenhart's Colloquium Lectures. By J. M. THOMAS ... 264
Three Books on Differential Equations. By W. R. LONGLEY 267
Cournot on Mathematical Economics. By G. C. EVANS 269
Three Books on Non-Euclidean Geometry. By E. S. ALLEN 271
Shorter Notices 277
Notes 279
New Publications 284

For official Communications and Notices, see the inside of the back cover.